隐函数

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一、定义.

$\mathbb{R}^2$中的点满足 $F(x,y)=0$ 的点，形成 $\mathbb{R}^2$ 中的曲线 / 函数关系.

例如 $ax+by+c=0$（直线），
$y=-\frac{1}{b}(ax+c)$ / $x=-\frac{1}{a}(by+c)$.
$F(x,y)=x^2+y^2-1=0$（圆），
$y=\sqrt{1-x^2}$, $y=-\sqrt{1-x^2}$. $x=\sqrt{1-y^2}$, $x=-\sqrt{1-y^2}$. 求导.

二、微分

1. 例. $y=e^{xy}$，求 $y'(x)$

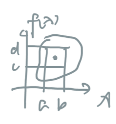
$y(x)=e^{xy(x)}$
$y'(x)=e^{xy(x)}(y(x)+xy'(x))$，$y'=\frac{ye^{xy}}{1-xe^{xy}}$.

若 $F(x,y)=0$ 给出函数关系，设 $y=y(x)=f(x)$.
$(x,f(x))$ 满足方程 $F(x,f(x))=0$.
两边对 x 求导，$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial f(x)}=F_x'+F_{f(x)}'f'(x)=0$.
$\therefore f'(x)=-\frac{F_x'(x,f(x))}{F_y'(x,f(x))}$

2. 定理.

设 $F(x,y)$ 是 $D\subset\mathbb{R}^2$ 中的二元函数，满足
① $F(x,y)\in C'(D)$（有连续偏导），
② 存在 $(x_0,y_0)\in D$，使 $F(x_0,y_0)=0$，
③ 且 $F_y'(x_0,y_0)\neq 0$ 或 $F_x'(x_0,y_0)\neq 0$，
则 $F(x,y)=0$ 在 (x_0,y_0) 的邻域内给出隐函数 $y=f(x)$ / 在 (x_0,y_0) 邻域内 $\exists\, y=f(x)$ 使得 $F(x,f(x))=0$. 且有 $y_0=f(x_0)$.
同时，$y=f(x)$ 有连续微商 $\frac{dy}{dx}=\frac{F_x'(x,y)}{F_y'(x,y)}$.

证明：1) $F_y'(x_0,y_0)\neq 0$，不妨设 $F_y'(x_0,y_0)>0$.



$\therefore \exists (x_0,y_0)$ 的邻域 $(a',b')\times(c,d)$
使 $F_y'(x,y)>0$
$\therefore$ 对 $\forall x\in[a,b]$，$F(x,y)$ 关于 y 严格单增.
$\therefore$ 有 $F(x_0,c)<F(x_0,y_0)<F(x_0,d)$
又 $F(x_0,y_0)=0$，$\therefore F(x_0,c)<0$，$F(x_0,d)>0$.
$\therefore F(x,y)$ 在 x_0 的邻域 $[a,b]$ 内，
有 $F(x,c)<0$，$F(x,d)>0$.
$\therefore \exists$ 唯一 $y_x\in[c,d]$ 使 $F(x,y_x)=0$.
即 $\exists\, y=f(x)\in[c,d]$，$F(x,f(x))=0$.
$\therefore y=f(x)$ 是 $F(x,y)=0$ 的隐函数.
且满足 $f(x_0)=y_0$.

2) 要证可微，$\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ 存在.
对 $\forall x\in(a,b)$，取 Δx，使 $x+\Delta x\in(a,b)$，
记 $\Delta y=f(x+\Delta x)-f(x)$.
$\because (x,f(x))$，$(x+\Delta x,f(x+\Delta x))$ 满足方程
$F(x,f(x))=0$，$F(x+\Delta x,f(x+\Delta x))=0$.
即 $F(x,y)=0$，$F(x+\Delta x,y+\Delta y)=0$.
$\therefore 0=F(x+\Delta x,y+\Delta y)-F(x,y)$
$=F(x+\Delta x,y+\Delta y)-F(x+\Delta x,y)+F(x+\Delta x,y)-F(x,y)$
Lagrange. $=F_y'(x+\Delta x,y+\theta_1\Delta y)\Delta y+F_x'(x+\theta_2\Delta x,y)\Delta x$
$\therefore \Delta y=-\frac{F_x'(x+\theta_2\Delta x,y)}{F_y'(x+\Delta x,y+\theta_1\Delta y)}\Delta x$.
$\because F_y'(x+\Delta x,y+\theta_1\Delta y)\neq 0$，$\therefore |\Delta y|\le M|\Delta x|$. （分母不为零，可放缩成常数.）
$\therefore |f(x+\Delta x)-f(x)|\le M|\Delta x|\to 0$，连续.
$\therefore \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}-\frac{F_x'(x+\theta_2\Delta x,y)}{F_y'(x+\Delta x,y+\theta_1\Delta y)}=-\frac{F_x'(x,y)}{F_y'(x,y)}$.
存在，则可微

3. 例. $\sin(x+y)+2x+y=0$

$F(x,y)=\sin(x+y)+2x+y$
$F_x'(x,y)=\cos(x+y)+2$，$F_y'(x,y)=\cos(x+y)+1$.
对 $y=f(x)$，$\frac{dy}{dx}=-\frac{F_x'(x,y)}{F_y'(x,y)}=-\frac{\cos(x+y)+2}{\cos(x+y)+1}$.
定义域 $\cos(x+y)+1\neq 0$，$x+y\neq(2n+1)\pi$

例. $F(x,y)=x^3+y^3-3axy$.

$F_x'=3x^2-3ay$，$F_y'=3y^2-3ax$.
$(0,0)$ 邻域内对 x, y 无隐函数.
$F_x'(0,0)=0$，$F_y'(0,0)=0$.
自相交：在交点 (x_0,y_0) 处，$F_x'=F_y'=0$.
取 $t=\frac{y}{x}$，$\begin{cases}x=x(t)=\frac{3at}{1+t^3}\\ y=y(t)=\frac{3at^2}{1+t^3}\end{cases}$

4. 应用.

① $F(x,y,z)=0$.
若 $z=f(x,y)$，$F(x,y,f(x,y))=0$
对 x 求导，$F_x'\cdot 1+F_y'\cdot 0+F_z'f_x'=0$.
对 y 求导，$F_x'\cdot 0$
$\frac{\partial f}{\partial x}=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)}$，$\frac{\partial f}{\partial y}=-\frac{F_y'(x,y,z)}{F_z'(x,y,z)}$.
$\therefore$ 存在，需求 $F_z'(x,y,z)\neq 0$.
或 $F_x'\neq 0$ 或 $F_y'\neq 0$，
$\therefore$ 只需 $\operatorname{grad}F\neq 0$，不是零向量.

② $F(x,y,z)=0$，$G(x,y,z)=0$.
设 $x=x$，$y=y(x)$，$z=z(x)$.
$\therefore \begin{cases}F_x'+F_y'y'+F_z'z'=0\\ G_x'+G_y'y'+G_z'z'=0\end{cases}$
有解，$\begin{vmatrix}F_y' & F_z'\\ G_y' & G_z'\end{vmatrix}=\frac{\partial(F,G)}{\partial(y,z)}\neq 0$
或 $\frac{\partial(F,G)}{\partial(x,y)}$ 或 $\frac{\partial(F,G)}{\partial(z,x)}\neq 0$.
$\therefore$ 只需 $\operatorname{grad}F\times\operatorname{grad}G\neq 0$，不共线.
（法向量不平行，平面有交线）.

③ $F(x,y,u,v)=0$，$G(x,y,u,v)=0$.
设 $u=u(x,y)$，$v=v(x,y)$.
$F(x,y,u(x,y),v(x,y))=0$，
$F_x'+F_u'u_x'+F_v'v_x'=0$，
$G_x'+G_u'u_x'+G_v'v_x'=0$.
存在，需要 $\frac{\partial(F,G)}{\partial(u,v)}\neq 0$.

二、逆映射.

$F(x,y,u,v)=x-x(u,v)=0$
$G(x,y,u,v)=y-y(u,v)=0$
$x=x(u,v)$，$y=y(u,v)$.
若存在逆映射，$v=v(x,y)$，$u=u(x,y)$.
$\frac{\partial u}{\partial x}=-\frac{\partial(F,G)}{\partial(x,y)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$
$=-\begin{vmatrix}1 & -x_v'\\ 0 & -y_v'\end{vmatrix}\Big/\begin{vmatrix}-x_u' & -x_v'\\ -y_u' & -y_v'\end{vmatrix}$

设 $F(x,y)=y-f(x)$.
$F(x,y)=0$，$y=f(x)$. 求 $x=g(y)$.
$F(g(y),y)=0$，$F_x'x_y'+F_y'=0$.
$\therefore x_y'=-\frac{F_y'}{F_x'}$.
即 $g'(y)=(f^{-1}(x))'=-\frac{1}{f'(x)}$

三、从微分看隐函数.

1. $F(x,y)\in C'(D)$. $dF(x,y)=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy$
若 $F(x,y)=0$ 确定 $y=f(x)$，
$F(x,f(x))=0$，$dF=(F_x'+F_y'f'(x))\,dx=0$.
$F_x'dx+F_y'dy=0$，$\frac{dy}{dx}=-\frac{F_x'}{F_y'}$.

2. $F(x_1,x_2,\dots,x_n)=0$
微分：$dF=F_{x_1}'dx_1+\dots+F_{x_n}'dx_n=0$.
若有 $F_{x_i}'\neq 0$，$dx_i=-\frac{1}{F_{x_i}'}(F_{x_1}'dx_1+\dots+F_{x_n}'dx_n)$
$\therefore \frac{\partial x_i}{\partial x_j}=-\frac{F_{x_i}'}{F_{x_j}'}$. （线性组合，组合系数为偏导）

3. 方程组 $\begin{cases}F_1(x_1,\dots,x_n)=0\\ F_2(x_1,\dots,x_n)=0\\ \vdots\\ F_m(x_1,\dots,x_n)=0\end{cases}$

微分得 $\begin{cases}dF_1=\frac{\partial F_1}{\partial x_1}dx_1+\dots+\frac{\partial F_1}{\partial x_n}dx_n=0\\ \vdots\\ dF_m=\frac{\partial F_m}{\partial x_1}dx_1+\dots+\frac{\partial F_m}{\partial x_n}dx_n=0\end{cases}$

不妨设 $m<n$，系数行列式 $\neq 0$，

$$\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m}\\ \vdots & & \vdots\\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_m}\end{pmatrix}\begin{pmatrix}dx_1\\ \vdots\\ dx_m\end{pmatrix}=\begin{pmatrix}-\frac{\partial F_1}{\partial x_{m+1}} & \cdots & \frac{\partial F_1}{\partial x_n}\\ \vdots & & \vdots\\ -\frac{\partial F_m}{\partial x_{m+1}} & \cdots & \frac{\partial F_m}{\partial x_n}\end{pmatrix}$$

$$\begin{pmatrix}dx_1\\ \vdots\\ dx_n\end{pmatrix}=A^{-1}B=C\begin{pmatrix}dx_{m+1}\\ \vdots\\ dx_n\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m}\\ \vdots & & \vdots\\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_m}\end{pmatrix}^{-1}\begin{pmatrix}-\frac{\partial F_1}{\partial x_{m+1}} & \cdots & \frac{\partial F_1}{\partial x_n}\\ \vdots & & \vdots\\ -\frac{\partial F_m}{\partial x_{m+1}} & \cdots & \frac{\partial F_m}{\partial x_n}\end{pmatrix}$$

隐函数.

1. $F(x,y)=0$. $F(x_0,y_0)=0$
隐函数 $y=f(x)$. $\therefore F(x,f(x))=0$.
对 x 求导. $F_x'(x,f(x))+F_y'(x,f(x))f'(x)=0$.
$\therefore f'(x)=-\frac{F_x'(x,f(x))}{F_y'(x,f(x))}=-\frac{F_x'(x,y)}{F_y'(x,y)}$
矩形邻域内. $F(x,y)$ 在 D 内有连续偏导.

例. $F(x,y)=\sin(x+y)+2x+y=0$.
求 $(x,y)=(0,0)$ 附近，$\frac{dy}{dx}$. $\frac{d^2y}{dx^2}$.
$F_x'(x,y)=\cos(x+y)+2$. $F_x'(0,0)=3$
$F_y'(x,y)=\cos(x+y)+1$. $F_y'(0,0)=2$.
F_y' 在 $(0,0)$ 的邻域内也不为零.
则 $\frac{dy}{dx}=-\frac{F_x'(x,y)}{F_y'(x,y)}=-\frac{\cos(x+y)+2}{\cos(x+y)+1}=-1-\frac{1}{\cos(x+y)+1}$
$\frac{d^2y}{dx^2}=\frac{-\sin(x+y)}{(\cos(x+y)+1)^2}\left(1+\frac{dy}{dx}\right)=\frac{\sin(x+y)}{[\cos(x+y)+1]^3}$
（可求切线）

例. 考虑方程 $F(x,y)=y^3-x=0$，在 $(0,0)$ 处
$F_y'(0,0)=0$，$F_x'(0,0)=0$. $\Rightarrow$ 非充要条件
但仍可求隐函数 $y=\sqrt[3]{x}$.

例. 考虑方程 $F(x,y)=x^3+y^3-3axy=0$.
可参数方程. $x=0$ 时，$y=0$.
设 $y=tx$，$x^3+t^3x^3-3atx^2=0$.
$(1+t^3)x=3at$，$t\neq -1$.
$\therefore x=\frac{3at}{t^3+1}$，$y=\frac{3at^2}{t^3+1}$. $(t\neq -1)$
$F_x'(x,y)=3x^2-3ay$. $F_x'(0,0)=0$
$F_y'(x,y)=3y^2-3ax$，$F_y'(0,0)=0$

交点多，原点处隐函数不存在.

n个变量 $F(x,y,z)=0$.
在 (x_0,y_0,z_0) 的邻域内有 $z=f(x,y)$.
$F(x,y,f(x,y))=0$ $\begin{cases}F_x'+F_z'\cdot f_x'=0\\ F_y'+F_z'f_y'=0.\end{cases}$
$\therefore f_x'=-\frac{F_x'}{F_z'}$，$f_y'=-\frac{F_y'}{F_z'}$.
只要 $(F_x',F_y',F_z')\neq(0,0,0)$（有一个非零）.
偏导连续. $F(x_0,y_0,z_0)=0$

2. ① 方程 $\begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases}$
在 (x_0,y_0,z_0) 的某个邻域有隐函数.
$y=y(x)$，$z=z(x)$. $F(x,y(x),z(x))$.
对 x 求导，得 $\begin{cases}F_x'+F_y'y'+F_z'z'=0\\ G_x'+G_y'y'+G_z'z'=0\end{cases}$
$\therefore y'(x)=\frac{\begin{vmatrix}-F_x' & F_z'\\ -G_x' & G_z'\end{vmatrix}}{\begin{vmatrix}F_y' & F_z'\\ G_y' & G_z'\end{vmatrix}}$　$z'(x)=\frac{\begin{vmatrix}F_y' & -F_x'\\ G_y' & -G_x'\end{vmatrix}}{\frac{\partial(F,G)}{\partial(y,z)}}$ (Crammer).
条件：$\frac{\partial(F,G)}{\partial(y,z)}$（Jacob 行列式 $\neq 0$）.
$\operatorname{grad}F\times\operatorname{grad}G=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z}\end{vmatrix}\neq 0$.

② 方程 $\begin{cases}F(x,y,u,v)=0\\ G(x,y,u,v)=0.\end{cases}$
设有隐函数 $u=u(x,y)$，$v=v(x,y)$.
$\begin{cases}F(x,y,u(x,y),v(x,y))=0\\ G(x,y,u(x,y),v(x,y))=0.\end{cases}$
对 x 求偏导 $\begin{cases}F_x'+F_u'u_x'+F_v'v_x'=0\\ G_x'+G_u'u_x'+F_v'v_x'=0.\end{cases}$
Crammer $\frac{\partial u}{\partial x}=\frac{-\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial v}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial v}\end{vmatrix}}{\frac{\partial(F,G)}{\partial(u,v)}}$　$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial u}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial u}\end{vmatrix}}{\frac{\partial(F,G)}{\partial(u,v)}}$

例. 设 $z=z(x,y)$ 是由方程 $e^z-xyz=0$ 确定的隐函数，求 $\frac{\partial z}{\partial x}$. $\frac{\partial z}{\partial y}$.
$F(x,y,z)=e^z-xyz=0$.
$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=-\frac{-yz}{e^z-xy}=\frac{yz}{e^z-xy}$.
$\frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=\dots$
$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{yz}{e^z-xy}\right)$
$=\frac{y\frac{\partial z}{\partial x}(e^z-xy)-yz(e^z\frac{\partial z}{\partial x}-y)}{(e^z-xy)^2}$

例. 设 $f(x,y)$ 是二阶连续可微函数，且 $\frac{\partial f}{\partial x}\neq 0$. 其中 $y=\varphi(x)$ 是 $f(x,y)=0$ 确定的隐函数，求 $\varphi''(x)$.
$\varphi'(x)=-\frac{F_x'}{F_y'}=-\frac{f_x'}{f_y'}$.
$\varphi''(x)=\frac{\partial}{\partial x}\left(-\frac{f_x'}{f_y'}\right)$
$f_{xy}''=f_{yx}''$.
$=-\frac{(f_{xx}''+f_{xy}''\varphi'(x))f_y'-f_x'(f_{yx}''+f_{yy}''\varphi'(x))}{(f_y')^2}$
$=-\frac{(f_{xx}''-f_{xy}''\cdot\frac{f_x'}{f_y'})f_y'-f_x'(f_{yx}''-f_{yy}''\cdot\frac{f_x'}{f_y'})}{(f_y')^2}$
$=-\frac{f_{xx}''(f_y')^2-2f_{xy}''f_x'f_y'+f_{yy}''(f_x')^2}{(f_y')^3}$.

例. $ax^2+by^2+cz^2-1=0$. $z=z(x,y)$.
对 x. $2ax+2cz\frac{\partial z}{\partial x}=0$. $\frac{\partial z}{\partial x}=-\frac{ax}{cz}$.
$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(-\frac{ax}{cz}\right)=-\frac{a}{c}\times\frac{z-x\frac{\partial z}{\partial x}}{z^2}=-\frac{acz^2+a^2x^2}{c^2z^3}$.

例. 设考虑变换 $u=u(x,y)$，$v=v(x,y)$ 有连续偏导数，且 $\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$ 证明
对二阶连续可微函数 $z=f(u,v)$ 恒成立 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$
证明：（观察式子，希望消去 v）.
① 先证 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.
$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)=0$.
② $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$
$=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}-\frac{\partial z}{\partial v}\cdot\frac{\partial u}{\partial y}$
$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}-\frac{\partial^2 z}{\partial u\partial v}\times\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\times\frac{\partial^2 u}{\partial x^2}$
将所有 $\frac{\partial v}{\partial x}$ 换成 $\frac{\partial u}{\partial y}$. $-\left(\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}-\frac{\partial^2 z}{\partial v^2}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y}-\frac{\partial z}{\partial v}\frac{\partial^2 u}{\partial x\partial y}$.
$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\times\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\times\frac{\partial v}{\partial y}$. $=\frac{\partial u}{\partial x}$.
$\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}$
$+\left(\frac{\partial^2 z}{\partial u\partial v}\times\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\times\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 u}{\partial x\partial y}$.
相加抵消. 合并. 因式分解.

3. 逆映射.

$\vec F: D\subset\mathbb{R}^{2n}\to\mathbb{R}^n$.　$(\vec x,\vec y)\to\vec F$
$(x_1,x_2,\dots,x_n),(y_1,\dots,y_n)\to\vec F=(F_1,\dots,F_n)$.
$\vec F(x,y)=\vec y-\vec f(\vec x)$，n 个为反函数.
$\vec x=\vec f^{-1}(\vec y)$

例如 $F(x,y,u,v)=u-\varphi(x,y)$
$G(x,y,u,v)=v-\psi(x,y)$
条件 $\frac{\partial(F,G)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\neq 0$
写作为 $\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}$
对 u 求偏导
$\begin{cases}\frac{\partial\varphi}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial\varphi}{\partial y}\times\frac{\partial y}{\partial u}=1\\ \frac{\partial\psi}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial\psi}{\partial y}\times\frac{\partial y}{\partial v}=0.\end{cases}$
$\therefore \frac{\partial x}{\partial u}=\frac{\frac{\partial v}{\partial y}}{\frac{\partial(u,v)}{\partial(x,y)}}$，$\frac{\partial y}{\partial u}=\frac{-\frac{\partial v}{\partial x}}{\frac{\partial(u,v)}{\partial(x,y)}}$.
$\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}\begin{pmatrix}\frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y}\\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}^{-1}$.

例. 极坐标变换的逆变换的偏微商.
变换：$x=x(r,\theta)=r\cos\theta$　$\Rightarrow r=r(x,y)$
$y=y(r,\theta)=r\sin\theta$.　$\theta=\theta(x,y)$
对 x 求偏导. $\cos\theta\frac{\partial r}{\partial x}-\sin\theta\, r\frac{\partial\theta}{\partial x}=1$
$\sin\theta\frac{\partial r}{\partial x}+r\cos\theta\frac{\partial\theta}{\partial x}=0$
解得 $\frac{\partial r}{\partial x}=\cos\theta$　同理. $\frac{\partial r}{\partial y}=\sin\theta$
$\frac{\partial\theta}{\partial x}=-\frac{\sin\theta}{r}$.　$\frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r}$
$\begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{pmatrix}^{-1}=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}^{-1}$
$=\frac{1}{r}\begin{pmatrix}r\cos\theta & r\sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$