微分

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一、偏导数

1. 定义：设 $f(x,y)$ 在 D 上有定义，$(x_0,y_0)\in D$，（仅考虑内点）

若 $\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}=a$，则称之关于 x 的偏导数，标为 $\frac{\partial f}{\partial x}(x_0,y_0)$.

$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=\frac{\partial f}{\partial x}(x_0,y_0)=f'_x(x_0,y_0)$

$\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}=\frac{\partial f}{\partial y}(x_0,y_0)$.

例. $f(x,y)=\sin xy$.

$\frac{\partial f}{\partial x}=y\cos xy$. y 看作常数.

例. $f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0).\end{cases}$

$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}$　$f(x,0)=0$，$f(0,y)=0$.

$=\lim_{\Delta x\to 0}\frac{\frac{2\Delta x\cdot 0}{\Delta x^2+0}-0}{\Delta x}=0$.

同理，$\frac{\partial f}{\partial y}(0,0)=0$.

「当 $x=y$ 时，$f(x,x)=\begin{cases}\frac{2x^2}{x^2+x^2}=1, & x\neq 0\\ 0, & x=0.\end{cases}$

在直线 $x=y$ 上不连续.

偏导数未必连续.」

2. 高阶导数

设 $f(x,y)$ 在 D 上有偏导.

$\frac{\partial f}{\partial x}(x,y)=f'_x(x,y)$，$\frac{\partial f}{\partial y}(x,y)=f'_y(x,y)$.

$\frac{\partial^2 f}{\partial x^2}=f''_{xx}(x,y)$. $\frac{\partial^2 f}{\partial y\partial x}=f''_{xy}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$

$\frac{\partial^2 f}{\partial y^2}=f''_{yy}(x,y)$. $\frac{\partial^2 f}{\partial x\partial y}=f''_{yx}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$.

定理：若 $\frac{\partial^2 f}{\partial x\partial y}$，$\frac{\partial^2 f}{\partial y\partial x}$ 连续，则相等.

证明：记 $\varphi(x)=f(x,y_0+\Delta y)-f(x,y_0)$.

$\psi(y)=f(x_0+\Delta x,y)-f(x_0,y)$

$\varphi(x_0+\Delta x)-\varphi(x_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)-f(x_0,y_0+\Delta y)+f(x_0,y_0)$

$\psi(y_0+\Delta y)-\psi(y_0)=\varphi(x_0+\Delta x)-\varphi(x_0)$

$\varphi(x_0+\Delta x)-\varphi(x_0)=\varphi'(x_0+\theta_1\Delta x)\Delta x$　微分中值.

$\psi(y_0+\Delta y)-\psi(y_0)=\psi'(y_0+\theta_2\Delta y)\Delta y$　对 y 求导是 y 的偏导.

$=[f'_y(x_0+\Delta x,y_0+\theta_2\Delta y)-f'_y(x_0,y_0+\theta_2\Delta y)]\Delta y$.

$=f''_{yx}(x_0+\gamma_2\Delta x,y_0+\theta_2\Delta y)\Delta y\Delta x$.　微分中值.

同理可得，

$\varphi(x_0+\Delta x)-\varphi(x_0)=f''_{xy}(x_0+\theta_1\Delta x,y_0+\gamma_1\Delta y)\Delta y\Delta x$

$\therefore f''_{xy}(x_0+\theta_1\Delta x,y_0+\gamma_1\Delta y)=f''_{yx}(x_0+\gamma_2\Delta x,y_0+\theta_2\Delta y)$.

取极限令 $\Delta x\to 0$，$\Delta y\to 0$，

连续，则有 $f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$.

* $\frac{\partial^n f}{\partial x^m\partial y^{n-m}}$ 高阶混合偏导.

* 初等函数，多变量也连续.

二、可微.

1. 定义

单变量　对函数 $y=f(x)$

$\Delta y=f(x+\Delta x)-f(x)=A\Delta x+o(\Delta x)$

则 $f(x)$ 在 x 可微，记 $df=A\Delta x$.

$df=Adx$，$A=f'(x)$. $f'(x)=\frac{df}{dx}$

多变量　对 $z=f(x,y)$，$(x,y)\in D\subset\mathbb{R}^2$.

$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$

$\triangleq a\Delta x+b\Delta y+o(\rho)$　（线性）

其中 $\rho=\sqrt{\Delta x^2+\Delta y^2}$，

则称 $z=f(x,y)$ 在 (x,y) 可微.

记为 $dz=df=A\Delta x+B\Delta y$.

$=Adx+Bdy$.

$\Delta y=0$，$f(x+\Delta x,y)-f(x,y)=a\Delta x+o(\Delta x)$

$\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=a+\frac{o(\Delta x)}{\Delta x}$.

$\Delta x=0$，$f(x\quad)-f(x,y)=\quad+o(\quad)$

$\frac{f(\quad)-f(x,y)}{\quad}=\quad+\frac{o(\quad)}{\quad}$

即可微时，一定存在 $x.y$ 的偏导，

且 $\frac{\partial f}{\partial x}(x,y)=a$，$\frac{\partial f}{\partial y}(x,y)=b$.

2. 定理

① 若 $f(x,y)$ 在 (x,y) 可微，则在 (x,y) 必有偏导，且 $df(x,y)=\frac{\partial f}{\partial x}(x,y)dx+\frac{\partial f}{\partial y}(x,y)dy$.

② 若 $f(x,y)$ 在 (x,y) 可微，则 $f(x,y)$ 在 (x,y) 连续.

证明：可微，则 $\exists a,b$ 使

$f(x+\Delta x,y+\Delta y)-f(x,y)=a\Delta x+b\Delta y+o(\rho)$.

$|f(x+\Delta x,y+\Delta y)-f(x,y)|=|a\Delta x+b\Delta y+o(\rho)|$

$\leq|a||\Delta x|+|b||\Delta y|+o(\rho)$

$\therefore\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}f(x+\Delta x,y+\Delta y)=f(x,y)$.

③ 设 $f(x,y)$ 在 D 上有偏导，要证 $f(x,y)$ 连续.

$|f(x+\Delta x,y+\Delta y)-f(x,y)|$

$=|f(x+\Delta x,y+\Delta y)-f(x+\Delta x,y)+f(x+\Delta x,y)-f(x,y)|$

$\leq|f(x+\Delta x,y+\Delta y)-f(x+\Delta x,y)|+|f(x+\Delta x,y)-f(x,y)|$

$=|f'_y(x+\Delta x,y+\theta_1\Delta y)||\Delta y|+|f'_x(x+\theta_2\Delta x,y)||\Delta x|$.

则当偏导有界时，

$\leq M|\Delta y|+M|\Delta x|\to 0$. 连续.

④ 设 $f(x,y)$ 在 D 上有偏导，要 $f(x,y)$ 可微，

$\frac{f(x+\Delta x,y+\Delta y)-f(x,y)-f'_x(x,y)\Delta x-f'_y(x,y)\Delta y}{\rho}$

$\leq\frac{|f'_y(x+\Delta x,y+\theta_1\Delta y)-f'_y(x,y)\Delta y|+|f'_x(x+\theta_2\Delta x,y)-f'_x(x,y)\Delta x|}{\rho}$

$\leq|f'_y(x+\Delta x,y+\theta_1\Delta y)-f'_y(x,y)|+|f'_x(x+\theta_2\Delta x,y)-f'_x(x,y)|$

则当偏导连续时，

$<\varepsilon$. $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(x+\Delta x,y+\Delta y)-f(x,y)-f'_x(x,y)\Delta x-f'_y(x,y)\Delta y}{\rho}=0$.

即 $f(x,y)$ 可微.

可微 ⇒ 连续，可微 ⇒ 偏导；连续 ⇍ 偏导（不推）

连续偏导 ⇒ 可微 ⇒ 连续，可微 ⇒ 偏导存在；连续 ⇍ 偏导存在（不推）

3. 推广

设 $z=f(x_1,x_2,\cdots x_n)$.

$\frac{\partial f}{\partial x_i}=\lim_{\Delta x_i\to 0}\frac{f(x_1,x_2,\cdots x_i+\Delta x_i,\cdots x_n)}{\Delta x_i}$

$\frac{\partial^n f(x_1\cdots x_n)}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}$. $df=\frac{\partial f}{\partial x_1}dx_1+\cdots+\frac{\partial f}{\partial x_n}dx_n$.

三、方向导数与梯度

1. 定义：设 $f(x,y)$ 在 D 上有定义，$(x_0,y_0)\in D$，

$\vec e$ 是二维单位向量，$\vec e=(\cos\alpha,\cos\beta)$. $\alpha+\beta=\frac{\pi}{2}$.

直线 L. $x(t)=x_0+t\cos\alpha$，$y(t)=y_0+t\cos\beta$.

$f(x,y)-f(x_0,y_0)=f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)$

若 $\lim_{t\to 0^+}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}$ 存在.

则称该极限为 $f(x,y)$ 在 (x_0,y_0) 沿 $\vec e$ 的方向导数，记为 $\frac{\partial f}{\partial\vec e}$

例. $r(x,y)=\sqrt{x^2+y^2}$ 沿 $\vec e=(\cos\alpha,\sin\alpha)$ 的方向导数.

$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}$. $\frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$

$\frac{\partial r}{\partial\vec e}=\frac{\partial r}{\partial x}\cos\alpha+\frac{\partial r}{\partial y}\sin\alpha=\frac{x}{\sqrt{x^2+y^2}}\cos\alpha+\frac{y}{\sqrt{x^2+y^2}}\sin\alpha$.

$=\sin\beta\cos\alpha+\cos\beta\sin\alpha=\cos(\beta-\alpha)$.

2. 定理.

设 f 在 D 上可微，则 f 沿任意方向 $\vec e$ 的方向导数都存在，且 $\frac{\partial f}{\partial\vec e}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta$.

证明：可微，$f(x+\Delta x,y_0+\Delta y)-f(x,y)$

$=f'_x(x,y)\Delta x+f'_y(x,y)\Delta y$

$\therefore f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x,y)$

$=f'_xt\cos\alpha+f'_yt\cos\beta+o(t)$

$\therefore\frac{f(x_0+t\cos\alpha,y+t\cos\beta)-f(x,y)}{t}$

$=f'_x\cos\alpha+f'_y\cos\beta+\frac{o(t)}{t}$.

3. 设 $f(x,y,z)$ 是 $V\subset\mathbb{R}^3$ 上的函数，$\vec e=\cos\alpha\vec i+\cos\beta\vec j+\cos\gamma\vec k$.

若 $\lim_{t\to 0^+}\frac{f(x+t\cos\alpha,y+t\cos\beta,z+t\cos\gamma)-f(x,y,z)}{t}$

存在，则该极限为向量导数.

$\frac{\partial f}{\partial\vec e}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma=\mathrm{grad}f\cdot\vec e$.

其中 $\mathrm{grad}f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$.

称为 $f(x,y,z)$ 在 (x,y,z) 的梯度（矢量）.

例. $z=f(x,y)=\sqrt{x^2+y^2}$

$z^2=x^2+y^2$，$\vec e=\cos\alpha\vec i+\cos\beta\vec j$.

$\frac{\partial f}{\partial\vec e}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta$

$=\frac{1}{\sqrt{x^2+y^2}}(x\cos\alpha+y\cos\beta)$

四、一阶微分形式的不变性.

1. 定理：设 $z=f(\xi,\eta)$ 在 D 中可微，

$\xi=\varphi(x,y)$. $\eta=\psi(x,y)$ 在 Δ 中可微，

则 $z=f(\varphi(x,y),\psi(x,y))$ 在 Δ 中可微.

证明：$\Delta z=f(\xi+\Delta\xi,\eta+\Delta\eta)-f(\xi,\eta)$

$=c_1\Delta\xi+c_2\Delta\eta+o(\tilde\rho)$. $\tilde\rho=\sqrt{\Delta\xi^2+\Delta\eta^2}$.

$\Delta\xi=\varphi(x+\Delta x,y+\Delta y)-\varphi(x,y)$

$=a_1\Delta x+a_2\Delta y+o(\rho)$.

$\Delta\eta=\psi(x+\Delta x,y+\Delta y)-\psi(x,y)$

$=b_1\Delta x+b_2\Delta y+o(\rho)$.

[要证 $\Delta z=A\Delta x+B\Delta y+o(\rho)$]

而 $\Delta z=c_1(a_1\Delta x+a_2\Delta y+o(\rho))$

$+c_2(b_1\Delta x+b_2\Delta y+o(\rho))+o(\tilde\rho)$

$=(c_1a_1+c_2b_1)\Delta x+(c_1a_2+c_2b_2)+o(\rho)+o(\tilde\rho)$

$\tilde\rho=\sqrt{\Delta\xi^2+\Delta\eta^2}$，$\tilde\rho^2=\Delta\xi^2+\Delta\eta^2$.

$\frac{\tilde\rho^2}{\rho^2}=\left(\frac{a_1\Delta x+a_2\Delta y+o(\rho)}{\rho}\right)^2+\left(\frac{b_1\Delta x+b_2\Delta y+o(\rho)}{\rho}\right)^2\leq M$.

又 $\rho\to 0$ 时，$\Delta x\to 0$，$\Delta y\to 0$. $\therefore\Delta\xi\to 0$，$\Delta\eta\to 0$.

$\therefore\lim_{\rho\to 0}\frac{o(\tilde\rho)}{\rho}=\lim_{\rho\to 0}\frac{o(\tilde\rho)}{\tilde\rho}\cdot\frac{\tilde\rho}{\rho}\leq\lim_{\rho\to 0}|M|\cdot 0=0$.

$\therefore\Delta z=(c_1a_1+c_2b_1)\Delta x+(c_1a_2+c_2b_2)\Delta y+o(\rho)$.

其中 $a_1=\frac{\partial\xi}{\partial x}$，$a_2=\frac{\partial\xi}{\partial y}$，$b_1=\frac{\partial\eta}{\partial x}$，$b_2=\frac{\partial\eta}{\partial y}$.

$c_1=\frac{\partial f}{\partial x}=\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial f}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}$，$c_2=\frac{\partial f}{\partial y}$.

2. 推广：$z=f(y_1,\cdots y_n)$，$y_i=y_i(x_1\cdots x_n)$.

$dz=f'_{y_1}dy_1+\cdots+f'_{y_n}dy_n$

$=f'_{x_1}dx_1+\cdots+f'_{x_n}dx_n$.

其中 $f'_{x_i}=\frac{\partial f}{\partial x_i}=\frac{\partial f}{\partial y_1}\frac{\partial y_1}{\partial x_i}+\cdots+\frac{\partial f}{\partial y_n}\frac{\partial y_n}{\partial x_i}$.

3. 例. $u=f(\xi,\eta)$，$\xi=x+y$，$\eta=x^2+y^2$.

$u=f(x+y,x^2+y^2)$

$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial x}=\frac{\partial f}{\partial\xi}+\frac{\partial f}{\partial\eta}2x$.

$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial y}=\frac{\partial f}{\partial\xi}+\frac{\partial f}{\partial\eta}2y$.

例. 设 $F(\xi,\eta)$ 可微，求 $F(x-y,x+y)$ 对 $x.y$ 偏导.

$\frac{\partial F}{\partial x}=F'_1(x)'+F'_2(x)'=F_1+F_2$，$\frac{\partial F}{\partial y}=F'_1-F'_2$.

例. $f(x+t\cos\alpha,y+t\cos\beta,z+t\cos\gamma)$

$\frac{df(t)}{dt}=f'_1\cos\alpha+f'_2\cos\beta+f'_3\cos\gamma=\mathrm{grad}f\cdot\vec e$

例. $u=f(x,y,z)$，$y=\varphi(x,r)$，$z=\psi(x,y,r)$，

求 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial r}$.

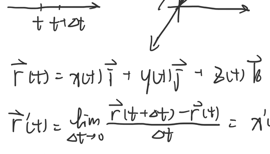
$u=f(x,\varphi(x,r),\psi(x,\varphi(x,r),r))$.

$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial\varphi}{\partial x}+\frac{\partial f}{\partial\psi}\times\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial x}\right)$

$\frac{\partial u}{\partial r}=\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial r}+\frac{\partial\psi}{\partial r}\right)$.

五、向量值函数.

1.



$\vec r(t)=x(t)\vec i+y(t)\vec j+z(t)\vec k$.

$\vec r'(t)=\lim_{\Delta t\to 0}\frac{\vec r(t+\Delta t)-\vec r(t)}{\Delta t}=x'(t)\vec i+y'(t)\vec j+z'(t)\vec k$.

2. 运算.

$\frac{d}{dt}(f(t)\vec a(t))=f'(t)\vec a(t)+f(t)\vec a'(t)$

$\frac{d}{dt}(\vec a(t)\cdot\vec b(t))=\vec a'(t)\cdot\vec b(t)+\vec a(t)\cdot\vec b'(t)$，$\times$ 同.

3. 例. $\vec r(t)=x(t)\vec i+y(t)\vec j+z(t)\vec k$，$|\vec r(t)|=c$，求 $\vec r'(t)$.

$\vec r(t)\cdot\vec r(t)=c^2$

$\therefore\vec r'(t)\cdot\vec r(t)+\vec r(t)\cdot\vec r'(t)=0$，$\vec r'(t)\cdot\vec r(t)=0$.

例. $\vec r(t)=r(t)\vec e_r$. 极坐标.

$\vec F(t)=f(t)\vec e_r=m\ddot{\vec r}$　有心场.

$=m(\ddot r-r\dot\theta^2)\vec e_r+(r\ddot\theta+2\dot r\dot\theta)\vec e_\theta$.

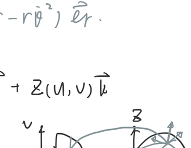
$r\ddot\theta+2\dot r\dot\theta=0$. $a=(\ddot r-r\dot\theta^2)\vec e_r$.

4. 设 $(u,v)\to(x,y,z)$

$\vec r(u,v)=x(u,v)\vec i+y(u,v)\vec j+z(u,v)\vec k$

$\frac{\partial\vec r}{\partial u}=\frac{\partial x}{\partial u}\vec i+\frac{\partial y}{\partial u}\vec j+\frac{\partial z}{\partial u}\vec k$.

$\frac{\partial\vec r}{\partial v}=\frac{\partial x}{\partial v}\vec i+\frac{\partial y}{\partial v}\vec j+\frac{\partial z}{\partial u}\vec k$.



设 $(u,v)\to(x,y)$

$x=x(u,v)$，$y=y(u,v)$.

$\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}$ Jacobi 行列式

设 $(u,v)\to(x,y)\to(\xi,\eta)$

$\xi=\xi(x,y)$，$\eta=\eta(x,y)$.

$\begin{pmatrix}\frac{\partial\xi}{\partial u}&\frac{\partial\xi}{\partial v}\\ \frac{\partial\eta}{\partial u}&\frac{\partial\eta}{\partial v}\end{pmatrix}=\begin{pmatrix}\frac{\partial\xi}{\partial x}&\frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x}&\frac{\partial\eta}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{pmatrix}$

$\begin{vmatrix}\frac{\partial\xi}{\partial u}&\frac{\partial\xi}{\partial v}\\ \frac{\partial\eta}{\partial u}&\frac{\partial\eta}{\partial v}\end{vmatrix}=\frac{\partial(\xi,\eta)}{\partial(u,v)}=\frac{\partial(\xi,\eta)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}$.

偏微商

1. 例. $f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

在原点处的偏导.

$f(x,0)=0$. $f(0,y)=0$

$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\frac{0-0}{x}=0$.

同理，$f'_y(0,0)=0$.

2. 高阶导数.

若函数 $f(x,y)=z$ 在区域 D 有定义.

如果 $\frac{\partial^2 f}{\partial x\partial y}$. $\frac{\partial^2 f}{\partial y\partial x}$ 连续，则两者相等.

证明：设 $M_0(x_0,y_0)\in D$，则 $\exists r>0$，使 $S(M_0,r)\subset D$.（内点）

记 $h=\Delta x\neq 0$，$k=\Delta y\neq 0$.

$0<|h|.|k|<r$ 时，$(x+h,y+k)\in S(M_0,r)\subset D$.

设 $\varphi(x)=f(x,y_0+k)-f(x,y_0)$，$\psi(y)=f(x_0+h,y)-f(x_0,y)$.　相近. f 可导. $\varphi.\psi$ 可导.

$\varphi(x_0+h)-\varphi(x_0)=f(x_0+h,y_0+k)-f(x_0+h,y_0)-f(x_0,y_0+k)+f(x_0,y_0)$

$=\psi(y_0+k)-\psi(y_0)$.

由微分中值定理，（两次）.

$\varphi(x_0+h)-\varphi(x_0)=h\varphi'(x_0+\theta_1h)$

$=h[f'_x(x_0+\theta_1h,y_0+k)-f'_x(x_0+\theta_1h,y_0)]$

$=hkf''_{xy}(x_0+\theta_1h,y_0+\gamma_1k)$.

同理，$\psi(y_0+k)-\psi(y_0)=khf''_{yx}(x_0+\theta_2h,y_0+\gamma_2k)$.

$h.k$ 如何时，对比可得 $f''_{xy}(x_0+\theta_1h,y_0+\gamma_1k)=f''_{yx}(x_0+\theta_2h,y_0+\gamma_2k)$.

其中 $0<|h|.|k|<r$，$\theta_1.\theta_2.\gamma_1\gamma_2\in(0,1)$.

趋于十字架. 但极限与趋于中心点相同.（极限邻域范围）

取极限有 $f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$　趋于中心点

3. 微分

① 一元函数 $f(x)-f(x_0)=f'(x_0)(x-x_0)+o(x-x_0)$

$\Delta f(h)=f'(x)h$，$df(x)=f'(x)dx$.

二元函数 $f(x,y)=f(x_0,y_0)+a(x-x_0)+b(y-y_0)+o(\rho)$.

$\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}$，则 $f(x,y)$ 可微.

（邻域一个平面到面）.

$(\Delta f)(h,k)\to$ 函数值的改变量

$\Delta f(h,k)=f(x_0+h,y_0+k)-f(x_0,y_0)$.

$df(h,k)=ah+bk=adx+bdy$.

（取 $f(x,y)=x$. $f(x,y)=y$ 时得出）.

② 在 (x_0,y_0) 处，$a=\frac{\partial f}{\partial x}(x_0,y_0)$. $b=\frac{\partial f}{\partial y}(x_0,y_0)$.

证明：$f(x_0+h,y_0+k)-f(x_0,y_0)=ah+bk+o(\rho)$.

取 $k=0$. $h\neq 0$. $f(x_0+h,y_0)-f(x_0,y_0)=ah+o(h)$.

$\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=a+o(1)$.

$\therefore\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=\frac{\partial f}{\partial x}(x_0,y_0)=a$.

$f(x,y)$ 在 (x_0,y_0) 可微 ⇒ 连续.

证明：$f(x_0+h,y_0+k)-f(x_0,y_0)=\underbrace{ah+bk}_{\to 0}+\underbrace{o(\rho)}_{\to 0}$　连续.

③ 两个偏导数有界 → 原函数连续.

证明：有界 ⇒ $M>0$，使 $|f'_x|<M$，$|f'_y|<M$.

任取 $M(x,y)\subset D$，存在 $\delta>0$，$S(M,\delta)\subseteq D$.（内点）

取 $0<|h|.|k|<\delta$，（更小的子区域）.

$|f(x+h,y+k)-f(x,y)|\to 0$.

$=|f(x+h,y+k)-f(x+h,y)+f(x+h,y)-f(x,y)|$

$=|kf'_y(x+h,y+\theta_2k)+hf'_x(x+\theta_1h,y+k)|$

$\leq|kf'_y(x+h,y+\theta_2k)|+|h||f'_x(x+\theta_1h,y+k)|$.

$\leq M|k|+M|h|=M(|h|+|k|)\to 0$.

$\therefore\lim_{\substack{h\to 0\\ k\to 0}}f(x+h,y+k)=f(x,y)$，连续.

两个偏导数连续 → 原函数可微.

证明：可微 ⇔ $\underbrace{\alpha(h,k)}_{\text{余项，记为 }o(\rho)}=f(x+h,y+k)-f(x,y)-f'_x(x,y)h-f'_y(x,y)k$.

$=f(x+h,y+k)-f(x+h,y)-f'_y(x,y)k$

$+f(x+h,y)-f(x,y)-f'_x(x,y)h$.

$=k[f'_y(x+h,y+\theta_2k)-f'_y(x,y)]$

$+h[f'_x(x+\theta_1h,y)-f'_x(x,y)]$.

$\therefore\frac{\alpha(h,k)}{\rho}\leq\frac{|k|}{\rho}|f'_y(x+h,y+\theta_2k)-f'_y(x,y)|$

$+\frac{|h|}{\rho}|f'_x(x+\theta_1h,y)-f'_x(x,y)|$.

$\leq|f'_y(x+h,y+\theta_2k)-f'_y(x,y)|$

$+|f'_x(x+\theta_1h,y)-f'_x(x,y)|$.

连续. $h.k\to 0$，$\frac{\alpha(h,k)}{\rho}\to 0$.

方向导数.

1. 定义

① 对单位向量 $\vec e(\cos\alpha,\sin\alpha)$.

$M=(x,y)+t(\cos\alpha,\sin\alpha)$. → 在射线上.

$M_0(x,y)$　$\overrightarrow{M_0M}=t\vec e$，$t>0$.

$\lim_{M\to M_0}\frac{f(M)-f(M_0)}{|M_0M|}=\lim_{t\to 0^+}\frac{f(x+t\cos\alpha,y+t\sin\alpha)-f(x,y)}{t}$

$=\frac{\partial f}{\partial\vec e}(x,y)=\frac{\partial f}{\partial\vec e}(M_0)$.

取 $\vec e=\vec i/\vec j$. 与偏导相似（但 $t\to 0^+$）.

② 有向直线.

$\lim_{M\to M_0}\frac{f(M)-f(M_0)}{\overline{MM_0}}=\lim_{t\to 0}\frac{f(x+t\cos\alpha,y+t\sin\alpha)-f(x,y)}{t}$

取 $\vec e=\vec i/\vec j$. 与偏导一样.

2. 计算.

$\lim_{t\to 0}\frac{f(x+t\cos\alpha,y+t\sin\alpha)-f(x,y)}{t}$

$=\lim_{t\to 0}\frac{1}{t}\left[\frac{\partial f}{\partial x}t\cos\alpha+\frac{\partial f}{\partial y}t\sin\alpha+o(t)\right]$

$=\lim_{t\to 0}\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha+o(1)=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha$.

$=\mathrm{grad}f\cdot\vec e$，其中 $\mathrm{grad}f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=\nabla f$.

例. 考虑函数 $f(x,y)=\sqrt{x^2+y^2}$ 在 $(0,0)$ 处的方向导数.

设 $\vec e=(\cos\alpha,\sin\alpha)$.

$\frac{\partial f(0,0)}{\partial\vec e}=\lim_{t\to 0^+}\frac{f(t\cos\alpha,t\sin\alpha)-f(0,0)}{t}=\lim_{t\to 0^+}\frac{t}{t}=1$.　定义1. 极限.

$\frac{\partial f(0,0)}{\partial\vec e}=\lim_{t\to 0}\frac{f(t\cos\alpha,t\sin\alpha)-f(0,0)}{t}=\lim_{t\to 0}\frac{|t|}{t}=\pm 1$.　定义2. 无极限.

复合函数.

1. 定理.

设 $u=f(\xi,\zeta)$ 在区域 $\tilde D$ 中可微

$\xi=\varphi(x,y)$，$\zeta=\psi(x,y)$ 在 D 中可微，

则复合函数 $u=f(\varphi(x,y),\psi(x,y))$ 在 D 中可微.

$\vec g(x,y)=(\varphi(x,y),\psi(x,y))$
（向量值函数）

$D\xrightarrow{\vec g(x,y)}\tilde D\xrightarrow{f}\mathbb{R}$，$D\xrightarrow{f\circ\vec g}\mathbb{R}$

证明：$\Delta\xi=\varphi(x+\Delta x,y+\Delta y)-\varphi(x,y)=a_1\Delta x+a_2\Delta y+o(\rho)$.

$\Delta\zeta=\psi(x+\Delta x,y+\Delta y)-\psi(x,y)=b_1\Delta x+b_2\Delta y+o(\rho)$.

$\Delta u=f(\xi+\Delta\xi,\zeta+\Delta\zeta)-f(\xi,\zeta)=c_1\Delta\xi+c_2\Delta\zeta+o(\tilde\rho)$.

其中 $\rho=\sqrt{\Delta x^2+\Delta y^2}$，$\tilde\rho=\sqrt{\Delta\xi^2+\Delta\zeta^2}$.

$\therefore\Delta u=c_1(a_1\Delta x+a_2\Delta y+o(\rho))+c_2(b_1\Delta x+b_2\Delta y+o(\rho))+o(\tilde\rho)$

$=(c_1a_1+c_2b_1)\Delta x+(c_1a_2+c_2b_2)\Delta y+o(\rho)+o(\tilde\rho)$

$\tilde\rho=\sqrt{[a_1\Delta x+a_2\Delta y+o(\rho)]^2+[b_1\Delta x+b_2\Delta y+o(\rho)]^2}$

当 $\rho\to 0$ 时，$\tilde\rho\to 0$.

且 $\frac{\tilde\rho^2}{\rho^2}\leq\left[|a_1|\frac{|\Delta x|}{\rho}+|a_2|\frac{|\Delta y|}{\rho}+o(1)\right]^2+\left[|b_1|\frac{|\Delta x|}{\rho}+|b_2|\frac{|\Delta y|}{\rho}+o(1)\right]^2$.

即当 $0<\rho<\delta$ 时，$\exists M$ 使 $\frac{\tilde\rho}{\rho}\leq M$. 有界.

$\therefore\lim_{\rho\to 0}\frac{o(\tilde\rho)}{\rho}=\lim_{\rho\to 0}\frac{o(\tilde\rho)}{\tilde\rho}\frac{\tilde\rho}{\rho}=0$.（从而 $o(\tilde\rho)$ 归入 $o(\rho)$）

$\therefore\Delta u=(c_1a_1+c_2b_1)\Delta x+(c_1a_2+c_2b_2)\Delta y+o(\rho)$.

$\therefore\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\zeta}\frac{\partial\zeta}{\partial x}$.

$=\frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial x}+\frac{\partial f}{\partial\psi}\frac{\partial\psi}{\partial x}$.

2. 一阶微分形式的不变性.

$u=f(\xi,\zeta)=f(\varphi(x,y),\psi(x,y))$

$du=\frac{\partial u}{\partial\xi}d\xi+\frac{\partial u}{\partial\zeta}d\zeta=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$

$=\left(\frac{\partial u}{\partial\xi}\frac{d\xi}{dx}+\frac{\partial u}{\partial\zeta}\frac{\partial\zeta}{dx}\right)dx+\left(\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial u}{\partial\zeta}\frac{\partial\zeta}{\partial y}\right)dy$.

更一般地，$u=f(\xi_1,\cdots\xi_m)$. $\xi_i=\varphi_i(x_1,\cdots x_n)$

$\frac{\partial u}{\partial x_j}=\sum_{i=1}^{m}\frac{\partial u}{\partial\xi_i}\frac{\partial\xi_i}{\partial x_j}$. $j=1\cdots n$.

映射 $\vec f:\mathbb{R}^n\to\mathbb{R}^m$.

$\vec x=(x_1,\cdots x_n)$. $\vec y=(y_1,\cdots y_m)=(f_1(\vec x),\cdots f_m(\vec x))$.

$d\vec y=d\vec f(\vec x)=(df_1(\vec x)\cdots df_m(\vec x))$.

$J_{\vec x}(\vec f)=\begin{pmatrix}\frac{\partial y_1}{\partial x_1}&\cdots&\frac{\partial y_1}{\partial x_n}\\ \vdots&&\\ \frac{\partial y_m}{\partial x_1}&\cdots&\frac{\partial y_m}{\partial x_n}\end{pmatrix}_{m\times n}$　$\vec f$ 的 Jacob 矩阵.

对复合映射 $\vec f:D\to D'$，$\vec g:D'\to\mathbb{R}^l$.

$(x_1,\cdots x_n)\xrightarrow{\vec f}(y_1,\cdots y_m)\xrightarrow{\vec g}(z_1,\cdots z_l)$.

$\frac{\partial z_i}{\partial x_j}=\sum_{k=1}^{m}\frac{\partial z_i}{\partial y_k}\cdot\frac{\partial y_k}{\partial x_j}$.

$\begin{pmatrix}\frac{\partial z_1}{\partial x_1}&\cdots&\frac{\partial z_1}{\partial x_n}\\ \vdots&&\vdots\\ \frac{\partial z_l}{\partial x_1}&\cdots&\frac{\partial z_l}{\partial x_n}\end{pmatrix}=\begin{pmatrix}\frac{\partial z_1}{\partial y_1}&\cdots&\frac{\partial z_1}{\partial y_m}\\ \vdots&&\vdots\\ \frac{\partial z_l}{\partial y_1}&\cdots&\frac{\partial z_l}{\partial y_m}\end{pmatrix}\begin{pmatrix}\frac{\partial y_1}{\partial x_1}&\cdots&\frac{\partial y_1}{\partial x_n}\\ \vdots&&\vdots\\ \frac{\partial y_m}{\partial x_1}&\cdots&\frac{\partial y_m}{\partial x_n}\end{pmatrix}$

$J(\vec g\cdot\vec f)(\vec x)=(J\vec g)(\vec f(\vec x))\times J\vec f(\vec x)$.

$m=l=n$ 时，有行列式.

例. $u=f(x,y,z)$，$y=\varphi(x,r)$，$z=\psi(x,y,r)$.

求 $\frac{\partial u}{\partial x}$. $\frac{\partial u}{\partial r}$.

$u=u(x,r)=f(x,\varphi(x,r),\psi(x,\varphi(x,r),r))$.

$\frac{\partial u}{\partial x}=f'_x+f'_y\varphi'_x+f'_z(\psi'_x+\psi'_y\varphi'_x)$.

$\frac{\partial u}{\partial r}=f'_y\varphi'_r+f'_z(\psi'_y\varphi'_r+\psi'_r)$　表示“第几位置”.

例. $u=\frac{q}{r}$，$r=\sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$.　$\left(\mathrm{grad}\,r=\frac{\vec r}{r}.\right)$

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x}=-\frac{q}{r^2}\times\frac{x}{r}=-q\frac{x}{r^3}$.

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(-q\frac{x}{r^3}\right)$

$=-q\frac{r^3-x\,3r^2\frac{x}{r}}{r^6}=-\frac{q}{r^3}+3q\frac{x^2}{r^5}$.

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3q}{r^3}+3q\frac{r^2}{r^5}=0$.

例. D 是空间凸区域（任意两点连线在 D 内）. 原点在 D 内.

设存在 D 内的可微函数 $f(x,y,z)$

满足 $xf'_x(x,y,z)+yf'_y(x,y,z)+zf'_z(x,y,z)=0$.

求证：$f(x,y,z)$ 恒为常数.

凸 开集 ⇒ 连通.

证明：任取 $(x,y,z)\in D$，$(x,y,z)\neq(0,0,0)$

令 $\varphi(t)=f(tx,ty,tz)$. $0\leq t\leq 1$.

原点到 (x,y,z) 直线段上的点.

易知 $\varphi(t)$ 在 $[0,1]$ 上连续.

当 $0<t<1$ 时，

$\varphi'(t)=xf'_x(tx,ty,tz)+yf'_y(tx,ty,tz)$

$+zf'_z(tx,ty,tz)=0$

$\therefore\varphi(t)$ 在 $[0,1]$ 上为常数.

特别有 $\varphi(1)=\varphi(0)$，即 $f(x,y,z)=f(0,0,0)$

例. 若函数 $f(x,y,z)$ 满足对 $\forall t>0$，$(x,y,z)\in\mathbb{R}^3$.

$f(tx,ty,tz)=t^kf(x,y,z)$，则称 $f(x,y,z)$ 为 k 次齐次函数. $k\in\mathbb{R}$.

求证：可微函数 $f(x,y,z)$ 为 k 次齐次函数

$\Leftrightarrow xf'_x(x,y,z)+yf'_y(x,y,z)+zf'_z(x,y,z)=kf(x,y,z)$

证明：(⇒) $f(tx,ty,tz)=t^kf(x,y,z)$ 对 t 求导.

$xf'_x(tx,ty,tz)+\cdots=kt^{k-1}f(x,y,z)$

取 $t=1$，$xf'_x(x,y,z)+\cdots=kf(x,y,z)$.

(⇐) [要证 $\frac{f(tx,ty,tz)}{t^k}=f(x,y,z)$]

令 $\varphi(t)=\frac{f(tx,ty,tz)}{t^k}$，

$\varphi'(t)=\frac{[xf'_x(tx,ty,tz)+\cdots]t^k-f(tx,ty,tz)kt^{k-1}}{t^{2k}}$

$=\frac{txf'_x(tx,\cdots)+\cdots-kf(tx,ty,tz)}{t^{k+1}}$

$=\frac{\tilde xf'_x(\tilde x,\tilde y,\tilde z)+\cdots-kf(\tilde x,\tilde y,\tilde z)}{t^{k+1}}=0$.

$\therefore\varphi(t)$ 恒为常数，特别 $\varphi(t)=\varphi(1)$

向量值函数.

1. $\vec r'(t)=\frac{d\vec r(t)}{dt}=\lim_{\Delta t\to 0}\frac{\vec r(t+\Delta t)-\vec r(t)}{\Delta t}$

$=\lim_{\Delta t\to 0}\left(\frac{x(t+\Delta t)-x(t)}{\Delta t},\cdots,\cdots\right)$.

$=(x'(t),y'(t),z'(t))$

2. $\frac{d}{dt}(\vec a\times\vec b)=\frac{d\vec a}{dt}\times\vec b+\frac{d\vec b}{dt}\times\vec a$.

证明：$\vec a(t)=(x_1(t),y_1(t),z_1(t))$

$\vec b(t)=(x_2(t),y_2(t),z_2(t))$.

$\frac{d}{dt}(\vec a\times\vec b)\cdot\vec i=\frac{d}{dt}(y_1(t)z_2(t)-y_2(t)z_1(t))$

$=y_1'z_2+y_1z_2'-y_2'z_1-y_2z_1'$

$=(y_1'z_2-y_2z_1')+(y_1z_2'-y_2z_1)=\left(\frac{d}{dt}\vec a\right)\times\vec b\cdot\vec i$.

3. $\vec r(u,v)=x(u,v)\vec i+y(u,v)\vec j+z(u,v)\vec k$.

$\frac{\partial\vec r}{\partial u}=\frac{\partial x}{\partial u}\vec i+\frac{\partial y}{\partial u}\vec j+\frac{\partial z}{\partial u}\vec k$.

$\frac{\partial\vec r}{\partial v}=\frac{\partial x}{\partial v}\vec i+\frac{\partial y}{\partial v}\vec j+\frac{\partial z}{\partial v}\vec k$.

$d\vec r(u,v)=dx(u,v)\vec i+dy(u,v)\vec j+dz(u,v)\vec k$

$=\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)\vec i+\cdots$

$=\left(\frac{\partial x}{\partial u}\vec i+\frac{\partial y}{\partial u}\vec j+\frac{\partial z}{\partial u}\vec k\right)du+\cdots$

$=\frac{\partial\vec r}{\partial u}du+\frac{\partial\vec r}{\partial v}dv$.