向量场的微分

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一、向量场的微分

1. 定义：在 $\mathbb{R}^3$ 中区域 V 上定义了向量值函数 $(x,y,t) \longrightarrow \vec{v}(x,y,t) = P(x,y,z)\vec{i} + Q\vec{j} + R\vec{k}$

2. 向量场的偏微商

$$\frac{\partial \vec{v}}{\partial x} = \lim_{\Delta x \to 0} \frac{\vec{v}(x+\Delta x, y, t) - \vec{v}(x,y,t)}{\Delta x}$$
$$= \frac{\partial P}{\partial x}\vec{i} + \frac{\partial Q}{\partial x}\vec{j} + \frac{\partial R}{\partial x}\vec{k}.$$

同理可得 $\frac{\partial \vec{v}}{\partial y}$, $\frac{\partial \vec{v}}{\partial z}$.

数段 $\varphi(x,y,z)$.

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = \mathrm{grad}\,\varphi \cdot d\vec{r}$$

其中 $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$，$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\mathrm{grad}\,\varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k}.$$

二、一些概念

1. 梯度.　数乘

记算符 $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

$\nabla\varphi = \varphi_x'\vec{i} + \varphi_y'\vec{j} + \varphi_z'\vec{k} = \mathrm{grad}\,\varphi$.

$d\varphi = \nabla\varphi\, dr$.　函数（数量场）→ 向量场.

2. 散度（divergence）　点乘

设 $\vec{r} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

定义 $\mathrm{div}\,\vec{v} = \nabla\cdot\vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

3. 旋度（Rotation）.　叉乘.

设向量场 $\vec{v} = \vec{v}(x,y,z)$,

$\mathrm{rot}\,\vec{v} = \nabla\times\vec{v} = (R_y' - Q_z')\vec{i} + (P_z' - R_x')\vec{j} + (Q_x' + P_y')\vec{k}$.

向量场 → 向量场.

$$\mathrm{rot}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

三、性质.

1. $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$.

$\nabla\cdot(\vec{a} + \vec{b}) = \nabla\cdot\vec{a} + \nabla\cdot\vec{b}$

$\nabla\times(\vec{a} + \vec{b}) = \nabla\times\vec{a} + \nabla\times\vec{b}$.

$\nabla(\varphi\psi) = \psi(\nabla\varphi) + \varphi(\nabla\psi)$

$\nabla\cdot(\varphi\vec{v}) = \nabla\varphi\cdot\vec{v} + \varphi\nabla\cdot\vec{v}$

$\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot\nabla\times\vec{a} - \vec{a}\cdot\nabla\times\vec{b}$

$\nabla\times(\varphi\vec{v}) = \nabla\varphi\times\vec{v} + \varphi\nabla\times\vec{v}$.

$\nabla\times\nabla\varphi = \mathrm{rot}(\mathrm{grad}\,\varphi) = 0$.

$\nabla\cdot(\nabla\times\vec{v}) = \mathrm{div}(\mathrm{rot}\,\vec{v}) = 0$.

2. Laplace 算符.

$\nabla\cdot\nabla = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \overset{记}{=} \Delta$.

$\Delta\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$

在区域 V 上，求 φ 满足 $\begin{cases} \Delta\varphi = 0 \\ \varphi|_{\partial D} = f(x,y,z) \end{cases}$?

四、∇ 在柱坐标下.

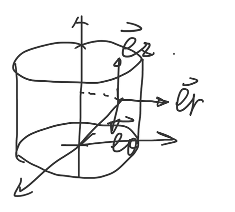
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$\vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j} + z\vec{k}$.　$(x,y,z) \to (r,\theta,z)$.

$\frac{\partial\vec{r}}{\partial r} = \cos\theta\vec{i} + \sin\theta\vec{j} = \vec{e_r}$.

$\frac{\partial\vec{r}}{\partial\theta} = -r\sin\theta\vec{i} + r\cos\theta\vec{j} = r\vec{e_\theta}$.

$\frac{\partial\vec{r}}{\partial z} = \vec{k} = \vec{e_z}$.



设 $\varphi = \varphi(x,y,z) = \varphi(r,\theta,z)$

$d\varphi = \varphi_x' dx + \varphi_y' dy + \varphi_z' dz$
$= \varphi_r' dr + \varphi_\theta' d\theta + \varphi_z' dz$　[一阶微分形式不变性.]

直角坐标　$\nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$.

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

柱坐标　$\nabla\varphi = a_1\vec{e_r} + a_2\vec{e_\theta} + a_3\vec{e_z}$.

$d\vec{r} = dr\vec{e_r} + rd\theta\vec{e_\theta} + dz\vec{e_z}$.

$d\varphi = \nabla\varphi\cdot d\vec{r} = (a_1\vec{e_r} + a_2\vec{e_\theta} + a_3\vec{e_z})\cdot(dr\vec{e_r} + rd\theta\vec{e_\theta} + dz\vec{e_z})$
$= a_1 dr + r a_2 d\theta + a_3 dz$.
$= \varphi_r' dr + \frac{1}{r}\varphi_\theta' d\theta + \varphi_z' dz$.

$\therefore \nabla\varphi = (\varphi_r'\vec{e_r} + \frac{1}{r}\varphi_\theta'\vec{e_\theta} + \varphi_z'\vec{e_z})\varphi$

$\nabla = \vec{e_r}\frac{\partial\varphi}{\partial r} + \frac{\vec{e_\theta}}{r}\frac{\partial\varphi}{\partial\theta} + \vec{e_z}\frac{\partial\varphi}{\partial z}$.

矢量分析.

1. 普通导数.　$df = \frac{df}{dx}dx$.

2. 梯度.　$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$.
$= \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$
$\cdot(dx\hat{x} + dy\hat{y} + dz\hat{z})$.
$= \nabla T\cdot d\vec{l}$

其中 $\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$ 为梯度.

3. 散度.　$\nabla\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(V_x\hat{x} + V_y\hat{y} + V_z\hat{z})$
$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$.

4. 旋度.　$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$
$= \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \cdots$

5. 运算性质.

$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$.　$\nabla(f+g) = \nabla f + \nabla g$,

$\frac{d}{dx}(kf) = k\frac{df}{dx}$.　$\nabla\cdot(A+B) = \nabla\cdot A + \nabla\cdot B$

$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$　$\nabla(kf) = k\nabla f$

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f\frac{dg}{dx} - g\frac{df}{dx}}{g^2}$　$\nabla kA = k\nabla A$.

$\nabla(fg) = f\nabla g + g\nabla f$.

$\nabla(A\cdot B) = A\times(\nabla\times B) + B\times(\nabla\times A) + (A\cdot\nabla)B + (B\cdot\nabla)A$.

$\nabla\cdot(fA) = f(\nabla\cdot A) + A\cdot(\nabla f)$.

$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$.

$\nabla\times(fA) = f(\nabla\times A) - A\times(\nabla f)$

$\nabla\times(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A)$

$\nabla(A\cdot B) = \frac{\partial\vec{A}\cdot\vec{B}}{\partial x} + \frac{\partial\vec{A}\cdot\vec{B}}{\partial y} + \frac{\partial\vec{A}\cdot\vec{B}}{\partial z}$

$=$

$\varepsilon_{ijk} = \begin{cases} 1 & i,j,k \\ -1 & j,i,k \\ 0 & \cdots \end{cases}$

$\varepsilon_{ijk} = -\varepsilon_{jik}$.　$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

$\nabla\Phi = (\partial_i\Phi)\hat{x}_i$

$\nabla\cdot\vec{F} = \partial_i F_i$,

$\nabla\times\vec{F} = \varepsilon_{ijk}\partial_j F_k\vec{e_i}$.

$\nabla\cdot(\nabla\times\vec{A}) = 0$, $\nabla\times\nabla\varphi = 0$. → 拆分可证.

链式法则　$\nabla\Phi(r) = \nabla r\frac{d\Phi}{dr}$

$\nabla\cdot\vec{A}(r) = \nabla r\cdot\frac{d\vec{A}}{dr}$

$\nabla\times\vec{A}(r) = \nabla r\times\frac{d\vec{A}}{dr}$

对 $U = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$

$\vec{E} = -\nabla U = -\frac{1}{4\pi\varepsilon_0}\frac{\nabla(\vec{p}\cdot\vec{r})}{r^3} - \frac{1}{4\pi\varepsilon_0}(\vec{p}\cdot\vec{r})\nabla\left(\frac{1}{r^3}\right)$

$= -\frac{1}{4\pi\varepsilon_0}\frac{\partial_i(p_j r_j)\vec{e_i}}{r^3} + \frac{3\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^4}\nabla r$

$= -\frac{1}{4\pi\varepsilon_0}\frac{p_j\delta_{ij}\vec{e_i}}{r^3} + \frac{3\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^4}\frac{\vec{r}}{r}$　　$\partial_i r_j = \delta_{ij}$

$= -\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}}{r^3} + \frac{3(\vec{p}\cdot\vec{r})}{4\pi\varepsilon_0 r^5}\vec{r}$

微分形式.

微分：$d\varphi = \varphi_x' dx + \varphi_y' dy + \varphi_z' dz$.

一般情况 $w = A dx + B dy + C dz$

（可以抛开函数求导）.

一、微分形式的空间

1. 定义外积 $\wedge$.

$dx\wedge dy$, $dy\wedge dz$, $dz\wedge dx$.

满足 $dx\wedge dy = -dy\wedge dx$.

$dx\wedge dx = 0$.

2. dx, dy, dz. → $dx\wedge dy$. → $dx\wedge dy\wedge dz$.

生成 $w^1 = A dx + B dy + C dz$

$w^2 = D dx\wedge dy + E dy\wedge dz + F dz\wedge dx$

$w^3 = h\, dx\wedge dy\wedge dz$.

$w^0 = \varphi$.

二、外积.

1. 对两个 1-形式.

$w_{\vec{v}}^1 = A_1 dx + B_1 dy + C_1 dz$

$w_{\vec{u}}^1 = A_2 dx + B_2 dy + C_2 dz$.

$w_{\vec{v}_1}^1\wedge w_{\vec{v}_2}^1 = (B_1C_2 - C_1B_2)\, dy\wedge dz$
$+ (C_1A_2 - A_1C_2)\, dz\wedge dx$
$+ (A_1B_2 - A_2B_1)\, dx\wedge dy$.
$= w^1_{\vec{v}_1\times\vec{v}_2}$

2. $w_{\vec{v}}^1\wedge w_{\vec{u}}^2 = w^3_{\vec{v}\times\vec{u}}$

3. $w_{\vec{v}}^1\wedge w_{\vec{u}}^3 = 0$,　$w_{\vec{v}}^2\times w_{\vec{u}}^2 = 0$.

4. $w_{\vec{v}_1}^1\wedge w_{\vec{v}_2}^1\wedge w_{\vec{v}_3}^1 = w^1_{\vec{v}_1}(w^2_{\vec{v}_2\times\vec{v}_3})$
$= w^3_{\vec{v}_1\cdot\vec{v}_2\times\vec{v}_3}$

5. $w_\varphi^0\wedge w_\psi^n = w_{\varphi\psi}^n$

三、外微分.

1. 定义. d：p 次形式 → $p+1$ 形式.

其中 $d w_\varphi^0 \to d\varphi = w^1_{\nabla\varphi}$

$d w_{\vec{v}}^1 = d(A dx + B dy + C dz)$
$= dA\wedge dx + dB\wedge dy + dC\wedge dz$
$= w^2_{\nabla\times\vec{v}}$

$d w_{\vec{v}}^2 = dD\wedge dy\wedge dt + dE\wedge dz\wedge dx + dF\wedge dx\wedge dy$.
$= (D_x' + E_y' + F_z')\, dx\wedge dy\wedge dz$
$= (\nabla\cdot\vec{v})\, dx\wedge dy\wedge dz = w^3_{\nabla\cdot\vec{v}}$.

$d w_{\vec{v}}^3 = 0$.

2. Poincaré 引理.

证明：若 w 为 0 形式（函数 φ）

$dw = \varphi_x' dx + \varphi_y' dy + \varphi_z' dz$
$= w^1_{\nabla\varphi}$

$d(dw) = d w^1_{\nabla\varphi} = w^2_{\nabla\times\nabla\varphi} = 0$.

若 w 为 1 形式,

$w_1 = A dx + B y + C dz = w_{\vec{v}}^1$.

$d w_{\vec{v}}^1 = w^2_{\nabla\cdot\vec{v}}$

$d(d w_{\vec{v}}^1) = d w^2_{\nabla\cdot\vec{v}} = w^3_{\nabla\cdot(\nabla\times\vec{v})} = 0$.

若为 2、3 形式，升次后超出 3，为 0.

3. 定义：对于给定的 w^i ($i \geq 1$)，若有一个低一次的形式 θ 使 $d\theta = w$，则称 w 是恰当的微分形式.（θ 是 w 的原函）

4. 定理：w 是恰当的 $\Leftrightarrow$ $dw = 0$.

证明：恰当，$d\theta = w$，$d(d\theta) = dw = 0$.

$dw = 0$，$d(d\theta) = 0$，有 $d(\theta) = w$.

例. $w = y\,dx + z\,dy + x\,dz$.

$dw = dy\wedge dx + dz\wedge dy + dx\wedge dz \neq 0$.

$w = yz\,dx + zx\,dy + xy\,dz$

$dw = d(yz)\wedge dx + d(zx)\wedge dy + d(xy)\wedge dz$

$w = d\varphi$, $\varphi = xyz$.

5. 推广

$w^p = \sum_{1\leq i_1 \cdots \leq i_p \leq n} a_{i_1, i_2 \cdots i_p}(x_1, \cdots x_p)\, dx_{i_1}\wedge\cdots\wedge dx_{i_p}$.

$w^p\wedge w^q = w^{p+q}$.

$w^1 = \sum_{i=1}^n a_i\, dx_i$,　$w^2 = \sum_{1\leq i<j\leq n} a_{ij}\, dx_i\wedge dx_j$.

$V^p = \{w^p\}$，$dV^p \to V^{p+1}$.

旋度. 梯度. 散度.

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

例. $\mathrm{rot}\,\mathrm{grad}\,\varphi = \nabla\times\nabla\varphi = 0$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} = 0$$，φ 二阶连续时.

$\mathrm{div}\,\mathrm{rot}\,\vec{a} = \nabla\cdot(\nabla\times\nabla) = 0$.

$\nabla\times\vec{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \cdots\right)$

$\nabla\cdot(\nabla\times\vec{a}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \cdots = 0$.

正交基.

$h_k = \frac{\partial x}{\partial u_k}$.　$\vec{e}_k = \frac{1}{h_k}\times\frac{\partial x}{\partial u_k}$

e_k 组成规范正交基.