证明　重开

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一、一阶微分方程.

1. 分离变量法

$f(y)\,dy = f(x)\,dx$

换元①. $P(x,y)dx + Q(x,y)dy = 0$

$\frac{dy}{dx} = \varphi(\frac{y}{x}) = \varphi(u)$

$= \frac{dux}{dx} = u + x\frac{du}{dx}$.

换元②. $\frac{dy}{dx} = \varphi(ax+by+c) = \varphi(u)$

$\frac{du}{dx} = a + b\frac{dy}{dx} = a + b\varphi(u)$

2. 一阶线性微分方程.

① $\frac{dy}{dx} + P(x)y = 0$. 齐次

S_1 分离变量. $\frac{dy}{dx} = -P(x)y$, $\frac{dy}{y} = -P(x)dx$.

$\ln|y| = \int -p(x)dx + C$, $y = Ce^{\int -p(x)dx}$.

S_2 积分因子. $e^{\int p(x)dx}\underbrace{\frac{dy}{dx}}_{y'} + e^{\int p(x)dx}p(x)y = 0$

$\therefore (e^{\int p(x)dx}y)' = 0$, $e^{\int p(x)dx}y = C$.

② $\frac{dy}{dx} + p(x)y = Q(x)$ 非齐次.

S_1. $y = C(x)e^{-\int p(x)dx}$.

代入, $-C(x)e^{-\int p(x)dx}p(x) + C'(x)e^{-\int p(x)dx} + p(x)C(x)e^{-\int p(x)dx}$

$= C'(x)e^{-\int p(x)dx} = Q(x)$

$C'(x) = e^{\int p(x)dx}Q(x)$, $C(x) = \int e^{\int p(x)dx}Q(x)dx + C$

$y = e^{-\int p(x)dx}(\int e^{\int p(x)dx}Q(x)dx + C)$.

S_2 积分因子.

3. 贝努利方程.

$\frac{dy}{dx} + p(x)y = q(x)y^n$.

$(1-n)y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$

$\therefore \frac{d(y^{1-n})}{dx} + p(x)y^{1-n} = q(x)$

$y^{1-n} = e^{-\int p(x)dx}(\int e^{\int p(x)dx}Q(x)dx + C)$

4. 可降阶的微分方程.

① $F(x, y', y'') \to F(x, p, \frac{dp}{dx})$

② $F(y, y', y'') \to F(y, p, \frac{dp}{dx}) = (y, p, p\frac{dp}{dy})$.

$\frac{dp}{dx} = \frac{dp}{dy} \times \frac{dy}{dx}$

*对二阶线性微分方程.

齐次: $y'' + p(x)y' + q(x)y = 0$.

非齐次: $y'' + p(x)y' + q(x)y = f(x)$.

齐次必有一组线性无关解 $y_1(x), y_2(x)$. (特解)

知 $y_1(x)$, $y_2(x) = y_1(x)\int \frac{1}{y_1^2(x)}e^{-\int p(x)dx}dx$.

则齐通解 $\bar{y}(x) = C_1y_1(x) + C_2y_2(x)$.

非齐通解 $y(x) = \bar{y}(x) + y_0(x)$ (通+特)

故只需求非齐特解.

常数变易法 $y_0(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$.

其中 $\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x). \end{cases}$

二、二阶微分方程.

1. 基本定义.

① 非齐次线性方程 $y'' + p(x)y' + q(x)y = f(x)$

齐次 $y'' + p(x)y' + q(x)y = 0$

② 线性相关..

③ 函数组的Wronski行列式.

2. 性质定理 ⇒ 求齐次方程.

① $\begin{cases} y'' + p(x)y' + q(x)y = f(x) \\ y(x_0) = \alpha, y'(x_0) = \beta \end{cases}$ 解唯一.

② 解的叠加.

③ 齐次线性方程必有基本解组 $y_1(x), y_2(x)$ (此两解线性无关).

证明: 取两组数 $\alpha_1, \alpha_2, \beta_1, \beta_2$ $\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0$.

对初值 α_1, β_1. α_2, β_2 的方程解.

存在且唯一, $y_1(x), y_2(x)$ 在 x_0 处

的 $W(x_0) = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0$.

$\therefore y_1(x), y_2(x)$ 线性无关.

④ 若有两个解 $y_1(x), y_2(x)$. 线性相关 ⇔ Wronski行列式恒为零.

证明: 线性相关, $C_1y_1(x) + C_2y_2(x) = 0$ 有非零解.

$C_1y_1'(x) + C_2y_2'(x) = 0$

$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0$. 行列相关.

取 $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$. ($\forall x_0 \in I$)

$\therefore \exists C_1, C_2 \neq 0$ 使 $C_1y_1(x_0) + C_2y_2(x_0) = 0$

$C_1y_1'(x_0) + C_2y_2'(x_0) = 0$

设 $y(x) = C_1y_1(x) + C_2y_2(x)$ 为解.

有 $y(x_0) = 0$, $y'(x_0) = 0$.

$\therefore C_1y_1(x_0) + C_2y_2(x_0) = 0$

⑤ 齐次方程两个解 $y_1(x), y_2(x)$ 与 $W(x)$ 可表示为 $W(x) = W(x_0)e^{-\int p(t)dt}$

证明: $y = (y_1, y_2)$ 行向量.

$y'' + py' + qy = 0$

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y \\ y' \end{vmatrix}$

$W'(x) = \begin{vmatrix} y' \\ y' \end{vmatrix} + \begin{vmatrix} y \\ y'' \end{vmatrix} = \begin{vmatrix} y \\ y'' \end{vmatrix}$

$= \begin{vmatrix} y \\ -py' - qy \end{vmatrix} = \begin{vmatrix} y \\ -py' \end{vmatrix}$

$= -p(x)W(x)$

$\therefore W'(x) + p(x)W(x) = 0$, $W(x)e^{\int_{x_0}^{x}p(t)dt} = C$.

令 $x = x_0$, $W(x_0) = C$. 代入.

⑥ 线性无关解 $y_1(x), y_2(x)$, 其 $W(x)$ 处处不为零.

方程任意一解可用它们的线性表示

关系式 $y_2(x) = y_1(x)\int \frac{1}{y_1^2(x)}e^{-\int p(x)dx}dx$.

$\left(\frac{y_2(x)}{y_1(x)}\right)' = \frac{y_1y_2' - y_2y_1'}{y_1^2} = \frac{W(x)}{y_1^2} = \frac{e^{-\int p(x)dx}}{y_1^2}$

3. 常数变易法 ⇒ 求非齐次方程.

齐次: $y''(x) + p(x)y'(x) + q(x)y(x) = 0$.

特解 $y_1(x), y_2(x)$, 通解 $y(x) = C_1y_1(x) + C_2y_2(x)$.

非齐次: $y''(x) + p(x)y' + q(x)y = f(x)$

通解 $y(x) = C_1y_1(x) + C_2y_2(x) + \underbrace{y_0(x)}_{特解}$

特解 $y_0(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$.

$C_1(x), C_2(x)$ 满足 $\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x) \end{cases}$

证明: 定义内积 $C \cdot y = C_1y_1 + C_2y_2$

并令 C_1, C_2 满足 $C' \cdot y = C_1'(x)y_1 + C_2'(x)y_2 = 0$.

4. 二阶常系数齐次线性微分方程.

$y'' + py' + qy = 0$.

特征方程 $\lambda^2 + p\lambda + q = 0$ 的解为 λ_0

⇔ 原方程的解为 $e^{\lambda_0 x}$.

证明: $y^{(n)} = (e^{\lambda_0 x})^{(n)} = \lambda_0^n e^{\lambda_0 x}$

对 $y'' + py' + qy = 0$, $e^{\lambda_0 x}(\lambda^2 + p\lambda + q) = 0$

特征方程.	原方程.
相异实根 λ_1, λ_2.	$e^{\lambda_1 x}, e^{\lambda_2 x}$.
重根 λ_0	$e^{\lambda_0 x}, xe^{\lambda_0 x}$
共轭复根 $a \pm ib$.	$e^{ax}\sin bx, e^{ax}\cos bx$

$T(\lambda) = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2$.

$y'' - 2\lambda_0y' + \lambda_0^2y = 0$, $y_1 = e^{\lambda_0 x}$

$y_2(x) = xe^{\lambda_0 x}$.

$e^{a+ib} = e^{ax}(\cos bx + i\sin bx)$

$e^{a-ib} = e^{ax}(\cos bx - i\sin bx)$

$\frac{e^{a+ib} + e^{a-ib}}{2} = e^{ax}\cos bx$

$\frac{e^{a+ib} - e^{a-ib}}{2} = e^{ax}\sin bx$