补充题

2021年12月14日 星期二　上午8:24

1. (2) $y - xy' = 2y + 2x^2y$

$\cancel{y} - x\frac{dy}{dx} = y + 2x^2y$

$\frac{dy}{y} = -\frac{(1+x^2)dx}{x}$

(3) $\tan x\sin^2 y\,dx + \cos^2 x\cot y\,dy = 0$

$\tan x\sin^2 y\,dx = -\cos^2 x\cot y\,dy$

$\frac{dy}{dx} = \frac{\tan x}{-\cos^2 x} \cdot \frac{\cot y}{\sin^2 y}$, $\frac{\sin^2 y}{\cot y}dy = \frac{\tan x}{-\cos^2 x}dx$

$\frac{\sin^2 y}{\frac{\cos y}{\sin y}}dy = -\frac{\frac{\sin x}{\cos x}}{\cos^2 x}dx$. $\frac{\sin^3 y}{\cos y}dy = -\frac{\sin x}{\cos^3 x}dx$

$\frac{\tan x}{\cos^2 x}dx = -\frac{\cot y}{\sin^2 y}dy$, $\tan^2 x - \cot^2 y = C$

(4) $\frac{dy}{dx} = \frac{1}{xy}\frac{1}{\sin^2(xy^2)} - \frac{y}{2x}$.

令$u = xy^2$. $\frac{du}{dx} = (xy^2)' = y^2 + 2xy\frac{dy}{dx}$.

$\therefore \frac{du}{dx} = \frac{2}{\sin^2 u}$.

2. $(3x^2 + 2xy - y^2)dx + (x^2 - 2xy)dy = 0$.

$3x^2dx + (2xy\,dx + x^2dy) - (y^2dx + 2xy\,dy) = 0$

$d(x^3) + d(x^2y) - d(xy^2) = 0$

$d(x^3 + x^2y - xy^2) = 0$.

3. (1) $\int_0^x tf(t)dt + \int_0^{f(x)} g(t)dt = x^2e^x$

$xf(x) + g(f(x))f'(x)\ e^x(2x + x^2)$

$xf(x) + 1 = \cancel{x}e^x(x+2)$, $f(x) = \begin{cases} e^x(x+2) - \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$xf(x) + xf'(x) = 2xe^x + x^2e^x$.

$f(x) + f'(x) = 2e^x + xe^x$. $(e^x f(x))' = e^{2x}(x+2)$

(2) $\int_0^1 f(xt)dt = \frac{1}{2}f(x) + 1$.

令$u = xt$. $\int_0^1 f(u)dt = \frac{1}{2}f(\frac{u}{t}) + 1$

$\int_0^1 f(u)d(\frac{u}{x}) = \frac{1}{2}f(x) + 1$. $\Rightarrow \int_0^x f(u)d = x\frac{1}{2}f(x) + x$.

对x求导, $xf'(x) - f(x) + 2 = 0$.

(3) $f(x) = -1 + x + 2\int_0^x tf(x-t)f'(x-t)dt$

令$u = x - t$. $f(x) = -1 + x + 2\int_0^x (x-u)f(u)f'(u)du$

$= -1 + x + 2\int_0^x (x-u)f(u)f'(u)dx$

对x求导. $f'(x) = 1 + 2f(u)f'(u)du = 1 + \int_0^x f(u)\,d\,f(u)$

1. (4) $\frac{dy}{dx} = \frac{1}{xy}\frac{1}{\sin^2(xy^2)} - \frac{y}{2x}$,

令$u = xy^2$, $xy = \frac{u}{y}$, $\frac{y}{x} = \frac{y}{\frac{u}{y^2}} = \frac{y^3}{u}$. $x = \frac{u}{y^2}$

$\frac{dy}{d\frac{u}{y^2}} = \frac{y}{u} \times \frac{1}{\sin^2 u} - \frac{1}{2}\frac{y^3}{u}$ $\frac{dxy}{dx} = x\frac{dy}{dx} + y$

$\frac{dy}{d\frac{u}{y^2}} = \frac{1}{\frac{d\frac{u}{y^2}}{dy}} = \frac{1}{2y\frac{du}{dy}}$

$\frac{du}{dx} = \frac{dxy^2}{dx} = y^2 + 2y\cdot x\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{\frac{du}{dx} - y^2}{\cancel{2yx}} = \frac{1}{\cancel{xy}}\frac{2}{\sin^2 u} - \frac{y^2}{\cancel{2x}}$

$\frac{du}{dx} = \frac{2}{\sin^2 u}$, $\sin^2 u\,du = 2dx$

(3) $\tan x\sin^2 y\,dx + \cos^2 x\cot y\,dy = 0$

$\tan x\sin^2 y\,dx = -\cos^2 x\cot y\,dy$

$\frac{\tan x}{\cos^2 x}dx = -\frac{\cot y}{\sin^2 y}dy$ $(\cot y)' = (\frac{\cos y}{\sin y})' = \frac{-\sin^2 y - \cos^2 y}{\sin^2 y}$

$\tan x\,d\tan x = \cot y\,d\cot y$.

$\frac{1}{2}\tan^2 x = \frac{1}{2}\cot^2 y + C$

$\tan^2 x = \cot^2 y + C_0$

2. (1) $(y + \sqrt{x^2+y^2})dx - xdy = 0$.

$\frac{dx}{x} = \frac{dy}{y + \sqrt{x^2+y^2}}$. $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} = \frac{y}{x} + \sqrt{1 + (\frac{y}{x})^2}$

$u = \frac{y}{x}$, $\frac{dy}{dx} = \frac{dux}{dx} = x\frac{du}{dx} + u$

$x\frac{du}{dx}\cancel{+u} = \cancel{u} + \sqrt{1+u^2}$ $\frac{du}{\sqrt{1+u^2}} = \pm\frac{dx}{x}$ $(x \neq 0)$.

按x正负取正负

$\ln(u + \sqrt{1+u^2}) = \pm\ln|x| + C_1$

$y + \sqrt{x^2+y^2} = C_2x^2$.

$\sqrt{x^2+y^2} - y = C_3$.

$x > 0$, $\ln(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}) = \ln x + C_1$.

$\ln(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}) - \ln x = C_1$

$\ln(\frac{y}{x^2} + \frac{1}{x}\sqrt{1 + \frac{y^2}{x^2}}) = C_1$

$\frac{y}{x^2} + \frac{1}{x}\sqrt{1 + \frac{y^2}{x^2}} = e^{C_1} = C_2$

$y + \sqrt{x^2+y^2} = C_2x^2$.

$x < 0$. $\ln(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}) = -\ln x + C_1$

$\ln(y + \sqrt{x^2+y^2}) = C_1$

$y + \sqrt{x^2+y^2} = e^{C_1} = C_2$

$\sqrt{x^2+y^2} - y = C$. $\ln(\sqrt{x^2+y^2} - y) = C_1$

$\ln(x\sqrt{1 + (\frac{y}{x})^2} - \frac{y}{x}\cdot x) = C_1$

$\ln(\sqrt{1 + (\frac{y}{x})^2} - \frac{y}{x}) = -\ln(x) + C_1$.

(2) $\frac{dy}{dx} = \frac{1 - 3(x+y)}{1 + (x+y)} = \frac{1 - 3z}{1 + z} = \frac{-3 - 3z + 4}{1 + z} = -3 + \frac{4}{1+z}$.

$x + y = z$, $\frac{dz}{dx} = \frac{dx+y}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 1 = -3 + \frac{4}{1+z}$.

$\frac{dz}{dx} = -2 + \frac{4}{1+z}$ $\frac{dz}{dx} - 1 = \frac{1-3z}{1+z}$

$\frac{dz}{dx} = \frac{1 - 3z + 1 + z}{1+z} = \frac{2 - 2z}{1+z} = 2\frac{1-z}{1+z}$.

$\frac{1+z}{1-z}dz = 2dx$. $\frac{z+1}{z-1} = -\frac{z-1+2}{z-1} = -1 - \frac{2}{z-1}$

$\int -1 - \frac{2}{z-1}dz = \int 2dx + C$

$-z - 2\ln|z-1| = 2x + C_1$.

(3) $y' = \frac{2y - x - 5}{2x - y + 4}$

$\begin{cases} 4y - 2x - \cancel{10} = 0 \\ 2x - y + 4 = 0 \end{cases} \Rightarrow 3y - 6 = 0.\ y = 2$
$\cancel{x}\ 2x + 2 = 0.\ x = -1$.

$\frac{2y - 4 - x - 1}{2x + 2 - y + 2} = y' = \frac{2(y-2) - (x+1)}{2(x+1) - (y-2)} = \frac{2\tilde{y} - \tilde{x}}{2\tilde{x} - \tilde{y}}$

$\frac{dy}{dx} = \frac{2\frac{\tilde{y}}{\tilde{x}} - 1}{2 - \frac{\tilde{y}}{\tilde{x}}} = \frac{d\tilde{y}}{d\tilde{x}}$, $u = \frac{\tilde{y}}{\tilde{x}}$,

$\frac{d\tilde{y}}{d\tilde{x}} = \tilde{x}\frac{du}{d\tilde{x}} + u = \frac{2u - 1}{2 - u}$

$\tilde{x}\frac{du}{d\tilde{x}} = \frac{2u - 1 - 2u + u^2}{2 - u} = \frac{u^2 - 1}{2 - u}$.

$\frac{2-u}{u^2-1}du = \frac{1}{\tilde{x}}d\tilde{x}$.

$(\frac{2}{u^2-1} - \frac{u}{u^2-1})du = \frac{1}{\tilde{x}}d\tilde{x}$

(4) $(y^2 - 6x)y' + 2y = 0$.

$\frac{dy}{dx}(y^2 - 6x) + 2y = 0$

$y^2 - 6x + 2y\frac{dx}{dy} = 0$

$\frac{dx}{dy} + \frac{y}{2} - 3\frac{x}{y} = 0$. $x' - \frac{3}{y}x = -\frac{1}{2}y$.

$x = e^{-\int -\frac{3}{y}dy}(\int e^{\int -\frac{3}{y}dy}(-\frac{1}{2}y)dy + C)$

$= e^{3\ln y}(\int e^{-3\ln y}(-\frac{1}{2}y)dy + C)$

$= y^3(\frac{1}{y^3}(-\frac{1}{2}y)dy + C)$

$= y^3(\int -\frac{1}{2y^2}dy + C) = y^3(\frac{1}{2}\frac{1}{y} + C)$

(5) $(y^4 - 3x^2)dy + xy\,dx = 0$

1. $F(x) = f(x)g(x)$.

$f'(x) = g(x)$, $g'(x) = f(x)$, $f(0) = 0$, $f(x) + g(x) = 2e^x$.

$F'(x) = f'(x)g(x) + f(x)g'(x)$

$= g^2(x) + f^2(x)$.

$f'(x) + g'(x) = 2e^x = f(x) + g(x)$.

$4e^{2x} = f'(x)f(x) + g'(x)g(x) + f'(x)g(x) + f(x)g'(x)$.

$= 2f(x)g(x) + f^2(x) + g^2(x). = (f(x) + g(x))^2$

$4e^{2x} = 2F(x) + F'(x)$.

2. $\int_0^{f(x)} g(t)dt = x^2e^x$.   g(x)为f(x)反函数.

f(x)在$[0,+\infty)$可导, $f(0) = 0$

$g(f(x))f'(x) = e^x(2x + x^2)$

$xf'(x) = e^x(2x + x^2)$, $f'(x) = e^x(x+2)$. , $f(x) = \overset{(x+1)}{\cancel{x}}e^x + C$

$f(0) = 0$, $C = -1$.

$xy\,dx = (3x^2 - y^4)\,dy$

$\frac{dx}{dy} = \frac{3x^2-y^4}{xy} = 3\frac{x}{y} - \frac{y^3}{x}$

$\frac{x}{y} = u.\quad x = uy.\quad \frac{dx}{dy} = \frac{d\,uy}{dy} = y\frac{du}{dy} + u.$

$y\frac{du}{dy} + u = 3u - \frac{1}{u}y^2.$

$\frac{du}{dy} = \frac{2u}{y} - \frac{y}{u}\quad, v = \frac{u}{y}.\quad \frac{du}{dy} = y\frac{dv}{dy} + v.$

$y\frac{dv}{dy} + v = 2v - \frac{1}{v},\quad \frac{dv}{dy}\cdot y = v - \frac{1}{v}.$

$\frac{1}{v-\frac{1}{v}}\,dv = \frac{1}{y}\,dy. = \frac{v}{v^2-1}\,dv.$

(6) $xy'\ln x\sin y + (1 - x\cos y)\cos y = 0.$

$y'\ln x\sin y + (\frac{1}{x} - \cos y)\cos y = 0$

$\frac{y'\sin y}{\cos y} + \frac{\frac{1}{x} - \cos y}{\ln x} = 0$

$-x\ln x\frac{d\cos y}{dx} + (1 - x\cos y) = 0.$

3. $\int_0^x t f(t)\,dt + \int_0^{f(x)} g(t)\,dt = x^2 e^x.$

$f(0)=0.$ $g(x)$ 反函数. 求 $f(x)$

$x f(x) + \underline{g(f(x))}_{x.}\, f'(x) = e^x(2x + x^2) = x e^x(x+2).$

$y = f(x),\quad x = f^{-1}(y) = \cancel{g(x)}.\ g(y).$

$y = f^{-1}(x) = g(x)$

$x f(x) + x f'(x) = e^x(2x + x^2)$

$f(x) + f'(x) = e^x(x+2).$

$y + \frac{dy}{dx} = e^x(x+2)$

$y = e^{-\int 1\,dx}(\int e^{\int 1\,dx} e^x(x+2)\,dx + C)$

$= e^{-x}(\int e^x\cdot e^x(x+2)\,dx + C)$

$= e^{-x}(\frac{1}{2}\int(x+2)\,de^{2x} + C)$

$= e^{-x}(\frac{1}{2}[(x+2)e^{2x} - \int e^{2x}\,d(x+2)] + C)$

$= e^{-x}(\frac{1}{2}[(x+2)e^{2x} - \frac{1}{2}e^{2x}] + C)$

4. $\int_0^1 f(xt)\,dt = \frac{1}{2}f(x) + 1.$

$u = xt.\quad \int_0^1 f(u)\,d\frac{u}{x} = \frac{1}{2}f(x) + 1.$

$\int_0^x \frac{1}{x} f(u)\,du = \frac{1}{2}f(x) + 1$

$\frac{1}{x}\int_0^x f(u)\,du = \frac{1}{2}f(x) + 1.$

$\frac{1}{x}F(x) = \frac{1}{2}F'(x) + 1.$

$\frac{y}{x} = \frac{1}{2}\frac{dy}{dx} + 1.\quad \frac{dy}{dx} - \frac{2y}{x} = -2.$

$y = e^{-\int -\frac{2}{x}dx}(\int e^{\int -\frac{2}{x}dx}(-2)\,dx + C)$

$= e^{2\ln x}(-2\int e^{-2\ln x}\,dx + C)$

$= x^2(-2\int\frac{1}{x^2}\,dx + C)$

$= x^2(\frac{2}{x} + C) = 2x + Cx^2.$

$y' = f(x) = 2 + 2Cx.$

5. $f(x) = -1 + x + 2\int_0^x t f(x-t) f'(x-t)\,dt.$

$u = x - t.\quad f(x) = -1 + x + 2\int_0^x (x-u) f(u) f'(u)\,d(x-u)$

$f(x) = -1 + x + 2\int_0^{-x}(x-u) f(u) f'(u)\,du.$

$= x - 1 - 2\int_0^x (x-u) f(u)\,df(u)$

$= x - 1 - 2\int_0^x x f(u)\,df(u) + 2\int_0^x u f(u)\,df(u)$

$= x - 1 - x f^2(u)\Big|_0^x + \int_0^x u\,d f^2(u)$

$= x - 1 - x f^2(x) + x f^2(0) + u f^2(u)\Big|_0^x - \int_0^x f^2(u)\,du$

$= x - 1 - x f^2(x) + x f^2(0) + x f^2(x) - \int_0^x f^2(u)\,du.$

$= x - 1 + x f^2(0) - \int_0^x f^2(u)\,du.\quad f(0) = -1.$

$f'(x) = 1 + f^2(0) - f^2(x)\qquad f'(x) = f^2(x).$

6. $f(x+y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}\quad, f'(0) = 1.$

$x = y = 0.\quad f(0) = \frac{2f(0)}{1 + f(0)^2}.$

$f(0) = 0.\qquad 1 + f^2(0) = 2.\ f(0) = \pm 1.$

$f(x) = \frac{f(x) + f(0)}{1 + f(x)f(0)} = \frac{f(x) + 1}{1 + f(x)} = x0.\ f(0) = 0$

$f(0) = \frac{f(x) + f(-x)}{1 + f(x)f(-x)} = 0.\quad f(x) = -f(-x).$ 奇.

$f(2x) = \frac{2f(x)}{1 + f^2(x)}\ ,\ f'(2x) = \frac{2f'(x)(1 + f^2(x)) - 2f(x)\,2f(x)}{(1 + f^2(x))^2}$

$\frac{2(f^2(x)f'(x) - 2f^2(x) + f(x))}{(1 + f^2(x))^2} = \frac{2f'(x) - 4f^2(x) + 2f^2(x)f(x)}{(1 + f^2(x))^2}$

可导. $f'(x) = \lim_{\Delta x\to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{\frac{f(x) + f(\Delta x)}{1 + f(x)f(\Delta x)} - f(x)}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{f(x) + f(\Delta x) - f(x)(1 + f(x)f(\Delta x))}{\Delta x(1 + f(x)f(\Delta x))}$

$= \lim_{\Delta x\to 0}\frac{f(\Delta x) - f^2(x)f(\Delta x)}{\Delta x(1 + f(x)f(\Delta x))}$

$= \lim_{\Delta x\to 0}\frac{f(\Delta x)(1 - f^2(x))}{\Delta x(1 + f(x)f(\Delta x))}$

$= \lim_{\Delta x\to 0}\frac{f(\Delta x) - f(0)}{\Delta x}\cdot\frac{(1 - f^2(x))}{1 + f(x)f(\Delta x)}$

$= \underset{1}{f'(0)} \times \frac{1 - f^2(x)}{1 + f(x)f(0)} = 1 - f^2(x)$