证明の重开

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一、定积分

1、可积性

(1) 可积函数有界.

(2) 闭区间上连续函数可积.

(3) 闭区间上单调函数可积.

2、计算性质

(1) 线性性：$\int_a^b \alpha f(x)+\beta g(x)\,dx$
$=\alpha\int_a^b f(x)\,dx+\beta\int_a^b g(x)\,dx$

(2) 保序性：$f(x)\ge g(x)$, $\int_a^b f(x)\,dx\ge\int_a^b g(x)\,dx$

(3) 保号性：$f(x)\ge 0$ 且不恒为 0, $\int_a^b f(x)\,dx>0$.

(4) 估值不等式 $m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$

(5) 积分中值：$\exists\xi$, 使 $f(\xi)(b-a)=\int_a^b f(x)\,dx$.

$f(x)$ 连续, $\exists m, M$, 对 $\forall x\in[a,b]$, $m\le f(x)\le M$.

$\therefore m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$,

$m\le\frac{1}{b-a}\int_a^b f(x)\,dx\le M$.

又 $f(x)$ 连续, $\therefore\exists\xi\in[a,b]$, 使 $f(\xi)=\frac{1}{b-a}\int_a^b f(x)\,dx$.

推广：$f(x)$ 连续, $g(x)$ 不变号, 则 $\exists\xi$,
使 $\int_a^b f(x)g(x)=f(\xi)\int_a^b g(x)\,dx$.

$f(x)$ 连续, $g(x)$ 不变号, $m\le f(x)g(x)\le M$.

不妨设 $g(x)\ge0$, 则 $\int_a^b g(x)dx\ge0$.

$\therefore m\int_a^b g(x)\,dx\le\int_a^b f(x)g(x)\,dx\le M\int_a^b g(x)\,dx$

$\therefore m\le\cdots\le M$. $f(x)g(x)$ 连续, $\exists\xi$, 使..

(6) 计算定义：$\int_a^a f(x)\,dx=0$. $\int_a^b f(x)\,dx=-\int_b^a f$.

(7) 区间可加：$\int_a^b f=\int_a^c f+\int_c^b f$

(8) 绝对值：$\left|\int_a^b f(x)\,dx\right|\le\int_a^b|f(x)|\,dx$.

二、变限积分

1. 连续性：$f(x)$ 在 $[a,b]$ 上可积, 则对 $\forall c\in[a,b]$,
有 $\int_c^x f(t)\,dt\triangleq\varphi(x)$ 连续.

证明：[连续：对 $\forall\varepsilon>0$, $\exists\delta$, 当 $|x-x_0|<\delta$ 时, $|\varphi(x)-\varphi(x_0)|<\varepsilon$]

$f(x)$ 在 $[a,b]$ 可积, 则 $\exists M>0$, 对 $\forall x\in[a,b]$, $|f(x)|\le M$.

对 $\forall\varepsilon>0$, 取 $\delta=\frac{\varepsilon}{M}$, 当 $|x-x_0|<\delta$ 时,

$|\varphi(x)-\varphi(x_0)|=\left|\int_c^x f(t)\,dt-\int_c^{x_0}f(t)\,dt\right|$

$=\left|\int_{x_0}^x f(t)\,dt\right|\le\int_{x_0}^x|f(t)|\,dt\le\int_{x_0}^x M\,dt$

$=M|x-x_0|<M\cdot\frac{\varepsilon}{M}=\varepsilon$. $\therefore$ 连续.

2. 可导：$f(x)$ 在 x_0 处连续, 则 $\varphi(x)=\int_c^x f(t)\,dt$ 在 x_0 处可导, 且 $\varphi'(x_0)=f(x_0)$.

证明：$\varphi'(x_0)=f(x_0)\Longleftrightarrow\lim\limits_{x\to x_0}\frac{\varphi(x)-\varphi(x_0)}{x-x_0}=\lim\limits_{x\to x_0}f(x)$

$\Longleftrightarrow\lim\limits_{x\to x_0}\left(\frac{\varphi(x)-\varphi(x_0)}{x-x_0}-f(x_0)\right)=0$

$\Longleftrightarrow\lim\limits_{x\to x_0}\left(\frac{1}{x-x_0}\left(\int_c^x f(t)\,dt-\int_c^{x_0}f(t)\,dt\right)-f(x_0)\right)$

$=\lim\limits_{x\to x_0}\left(\frac{1}{x-x_0}\int_{x_0}^x f(t)\,dt-f(x_0)\right)$

$=\lim\limits_{x\to x_0}\left[\frac{1}{x-x_0}\left(\int_{x_0}^x f(t)\,dt-(x-x_0)f(x_0)\right)\right]$

$=\lim\limits_{x\to x_0}\frac{1}{x-x_0}\left(\int_{x_0}^x f(t)\,dt-\int_{x_0}^x f(x_0)\,dt\right)$

$=\lim\limits_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^x(f(t)-f(x_0))\,dt=0$

对 $\forall\varepsilon>0$, $\exists\delta$, 当 $|x-x_0|<\delta$ 时, $|f(x)-f(x_0)|<\varepsilon$.

$\therefore\lim\limits_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^x(f(t)-f(x_0))\,dt$

$<\lim\limits_{x\to x_0}\left|\frac{1}{x-x_0}\right|\int_{x_0}^x\varepsilon\,dt=\lim\limits_{x\to x_0}\frac{\varepsilon(x-x_0)}{x-x_0}=\varepsilon$.

1、可积性

(1) 定理

① 可积的充要判别.

$f(x)$ 在 $[a,b]$ 可积 $\Longleftrightarrow$

$\lim\limits_{\|T\|\to0}\sum\limits_{i=1}^n\omega_i\Delta x_i=0$. $\Longleftrightarrow\sum\limits_{i=1}^n\omega_i\Delta x_i<\varepsilon$.

$\Longleftrightarrow\inf\{\bar S(T)\}=\sup\{\underline S(T)\}$.

② 加密分割 T 为 T',

$\bar S(T)\ge\bar S(T')$, $\underline S(T)\le\underline S(T')$.

$\bar S(T)\ge\bar S(T')+l\omega\|T\|$. ($l$ 为增加的分割点)

③ 分割 $T_1\cup T_2=T$,

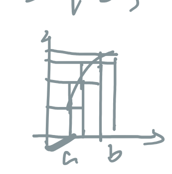
则 $\underline S(T_1)\le\underline S(T)\le\bar S(T)\le\bar S(T_2)$

④ 记 $\bar I=\inf\bar S(T)$, $\underline I=\sup\underline S(T)$, 则有

$\lim\limits_{\|T\|\to0}\bar S(T)=\bar I$, $\lim\limits_{\|T\|\to0}\underline S(T)=\underline I$.

证明：$\bar I=\inf\bar S(T)$, 则对 $\forall\varepsilon>0$, $\exists T_\varepsilon$,

有 $\bar I<\bar S(T_\varepsilon)\le\bar I+\frac{\varepsilon}{2}$.



对 $\forall$ 分割 T, 当 $\|T\|\le\frac{\varepsilon}{2l\omega+1}$,

有 $\bar I\le\bar S(T)\le\bar S(T\cup T_\varepsilon)+l\omega\|T\|$

$\le\bar S(T_\varepsilon)+l\omega\|T\|$

$<\bar I+\frac{\varepsilon}{2}+l\omega\left(\frac{\varepsilon}{2l\omega+1}\right)<\bar I+\varepsilon$.

(2) 充分条件判断

① 闭区间上的连续函数可积.

证明：设 $f(x)$ 在 $[a,b]$ 上连续, 则 $f(x)$ 一致连续.

$\therefore$ 对 $\forall\varepsilon>0$, $\exists\delta_\varepsilon$, 当 $|x_1-x_2|<\delta_\varepsilon$ 时,

$|f(x_1)-f(x_2)|<\frac{1}{b-a+1}\varepsilon$.

则 $\exists$ 分割 T, 当 $\|T\|<\delta_\varepsilon$ 时,

$\omega_i\le|f(x_i)-f(x_{i-1})|=\frac{1}{b-a+1}\varepsilon$,

$\sum\limits_{i=1}^n\omega_i\Delta x_i\le\sum\limits_{i=1}^n\frac{1}{b-a+1}\varepsilon(x_i-x_{i-1})$

$=\frac{\varepsilon}{b-a+1}(b-a)<\varepsilon$. $\therefore$ 可积.

② 有界闭区间上单调递增(减)的函数可积.

证明：1) 若 $f(x)$ 为常值函数, 显可积.

2) 若 $f(x)$ 不为常值, 不妨设有 $f(a)<f(b)$.

上, 有 $f(a)<f(b)$.

对 $\forall\varepsilon>0$, 取 $\delta_\varepsilon=\frac{\varepsilon}{f(b)-f(a)}$, 当 $\|T\|<\delta_\varepsilon$ 时,

有 $\bar S(T)-\underline S(T)=\sum\limits_{i=1}^n\omega_i\Delta x_i\le\sum\limits_{i=1}^n\omega_i\|T\|$

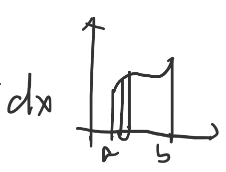
$<\sum\limits_{i=1}^n\omega_i\frac{\varepsilon}{f(b)-f(a)}=\sum\limits_{i=1}^n(f(x_i)-f(x_{i-1}))\frac{\varepsilon}{f(b)-f(a)}$

$=(f(b)-f(a))\cdot\frac{\varepsilon}{f(b)-f(a)}=\varepsilon$.

二、微分法与应用.

1. 弧长.

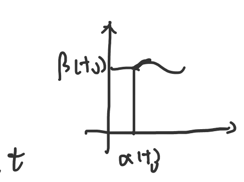
① 对 $y=f(x)$, $l=\int_a^b\sqrt{(\Delta x)^2+(f(x+\Delta x)-f(x))^2}\,dx$



$=\int_a^b\Delta x\sqrt{1+\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)^2}=\int_a^b\Delta x\sqrt{1+(f'(x))^2}\,dx$

$=\int_a^b\sqrt{1+(f'(x))^2}\,dx$.

② 对 $\vec\gamma=(\vec\alpha(t),\vec\beta(t))$,



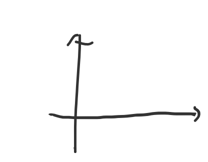
$l=\int_\alpha^\beta\sqrt{(\alpha(t+\Delta t)-\alpha(t))^2+(\beta(t+\Delta t)-\beta(t))^2}\,dt$

$=\int_\alpha^\beta\Delta t\sqrt{\left(\frac{\alpha(t+\Delta t)-\alpha(t)}{\Delta t}\right)^2+\left(\frac{\beta(t+\Delta t)-\beta(t)}{\Delta t}\right)^2}\,dt$

$=\int_\alpha^\beta\Delta t\sqrt{\alpha'(t)^2+\beta'(t)^2}\,dt$

$=\int_\alpha^\beta\sqrt{\alpha'(t)^2+\beta'(t)^2}\,dt=\int_\alpha^\beta|\vec\gamma'(t)|\,dt$.

③ 对 $r=r(\varphi)$



$\begin{cases}x=r(\varphi)\cos\varphi\\ y=r(\varphi)\sin\varphi.\end{cases}$

$l=\int_\alpha^\beta\sqrt{(r(\varphi)\cos\varphi)'^2+(r(\varphi)\sin\varphi)'^2}\,d\varphi$

$=\int_\alpha^\beta\sqrt{(r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi)^2+(r'(\varphi)\sin\varphi+r(\varphi)\cos\varphi)^2}\,d\varphi$

$=\int_\alpha^\beta\sqrt{r(\varphi)^2+r'(\varphi)^2}\,d\varphi$.

2. 面积

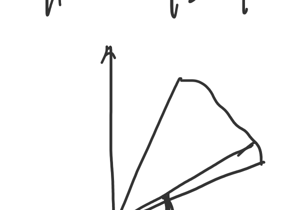
① $y_1=f_1(x)$. $y_2=f_2(x)$. 与 $x_1=a$. $x_2=b$.

$S=\int_a^b f_1(x)\,dx-\int_a^b f_2(x)\,dx$.

② 对 $r=r(\varphi)$ 与两条射线 $\varphi_1=\alpha$. $\varphi_2=\beta$.

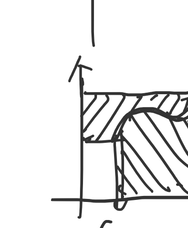
$S=\int_\alpha^\beta r(\varphi)$

$=\frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,d\varphi$



③ ~~$S=\int_a^b\frac{(f(x)+f(x+\Delta x))\Delta x}{2}dx$~~

~~$=\frac{1}{2}\int_a^b\Delta x$~~



$\vec r(t)=x(t), y(t)$.

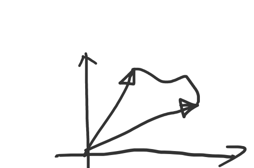
$S=\int_a^b y(t)(x(t+\Delta t)-x(t))\,dt$

$=\int_a^b y(t)x'(t)\,dt$.

$\Delta S=x(t)(y(t+\Delta t)-y(t))$

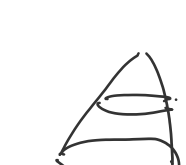
$S=\int_a^b x(t)y'(t)\,dt$.

④ $S=\frac{1}{2}\int_a^b(x(t)y'(t)-x'(t)y(t))\,dt$.



3. 体积.

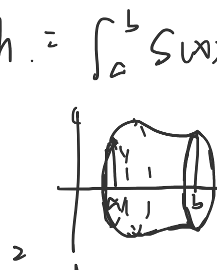
① 基本求法.

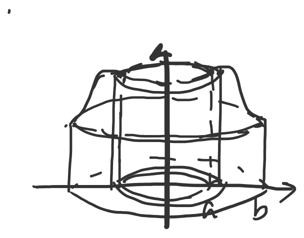


$\Delta V_h=S_h\cdot\Delta h$.

$V=\int_{h_1}^{h_2}S_h\,dh=\int_a^b S(x)\,dx$.

② 旋转体.



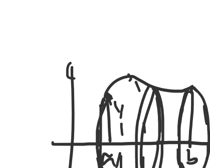


$V_1=\int_a^b\pi f^2(x)\,dx$

$\Delta V=\pi((x+\Delta x)^2-x^2)f(x)=2\pi x f(x)\,dx+\pi f(x)(dx)^2$

$V_2=2\pi\int_a^b dV=2\pi\int_a^b x f(x)\,dx$.

③ 侧面积.



圆台侧面积. $\pi(y+dy+y)\,ds$

$=2\pi y\,ds+\pi\,dy\,ds$.

$S=\int_a^b 2\pi|y(x)|\sqrt{1+y'^2(x)}\,dx$.