有理式的不定积分

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一、命题总结

1. 定理：连续函数 - 一定有原函数。
   初等函数 - 一定有原函数。
2. 方法：① 分部积分法
   ② 换元积分法

二、多项式（有理函数）

1. 定义：$P(x) = a_n x^n + \cdots + a_1 x + a_0$
   $Q(x) = b_n x^n + \cdots + b_1 x + b_0$
   假分式：分子次数 ≥ 分母次数
   真分式：分子次数严格小于分母。
   假分式 $\xrightarrow{\text{辗转相除法}}$ 多项式 + 真分式。
   证明：对 $\deg P(x) = m$，$\deg Q(x) = n$，$n < m$，
   一定存在多项式 $R(x)$，$\deg R(x) < m$，
   使 $P(x) = Q(x)D(x) + R(x)$，
   $\therefore \frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)}$。

例：$P(x) = x^5 + 3x + 1$，$Q(x) = 2x^2 + 3$。

$$\begin{array}{r} \frac{1}{2}x^3 + 0 - \frac{3}{4}x \\ 2x^2+3 \,\big)\, x^5 + 0x^4 + 0x^3 + 0x^2 + 3x + 1 \\ x^5 + 0 + \frac{3}{2}x^3 \\ \hline -\frac{3}{2}x^3 + 0x^2 + 3x + 1 \\ -\frac{3}{2}x^3 + 0 - \frac{9}{4}x \\ \hline \frac{21}{4}x + 1 \end{array}$$

$\Rightarrow P(x) = (2x^2+3)(\frac{1}{2}x^3 - \frac{3}{4}x) + \frac{21}{4}x + 1$

S_2. 待定系数。
$x^5 + 3x + 1 = (2x^2+3)(c_3x^3 + c_2x^2 + c_1x + c_0) + R(x)$
$\Rightarrow 2c_3 = 1$，$2c_2 = 0$，$2c_1 + 3c_3 = 0$ ← $\deg R(x)$
$\therefore c_3 = \frac{1}{2}$，$c_2 = 0$，$c_1 = -\frac{3}{4}$，$c_0 = 0$。

2. 多项式分解。
   如 $x^2 - 9$ 在整数中没有根，在 $\mathbb{Q}$ 中有根。
   $x^2 + 1$ 在 $\mathbb{R}$ 中无根，在 $\mathbb{C}$ 中有根。
   代数学基本定理：实系数多项式在 $\mathbb{C}$ 中可分解。
   $P(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$。
   其中 k 个是实根，$n-k = 2l$ 个是虚根，两两共轭
   对 $(x-\alpha)(x-\bar{\alpha}) = x^2 + \beta x + \gamma$，$\beta^2 - 4\gamma^2 < 0$。
   则有 $P(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x^2+\beta_1 x+\gamma_1)(x^2+\beta_2 x+\gamma_2)$
3. 有理式分解。
   $Q = (x-\alpha_1)^{r_1}(x-\alpha_2)^{r_2}\cdots(x^2+\beta_1 x+\gamma_1)^{s_1}(x^2+\beta_l x+\gamma_l)^{s_l}$
   设 $\frac{P(x)}{Q(x)}$ 是真分式，则

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{(x-\alpha_1)} + \cdots + \frac{A_{r}}{(x-\alpha_k)^{r_k}} + \frac{B_1x + C_1}{(x^2+\beta_1x+\gamma_1)} + \cdots + \frac{B_{1s_1}x + C_{1s_1}}{(x^2+\beta x+\gamma_1)^{s_1}} + \frac{B_{l1}x + C_1}{x^2+\beta_l x+\gamma_l} + \cdots + \frac{B_{ls_l}x + C_{ls_l}}{(x^2+\beta_l x+\gamma_l)^{s_l}}$$

三、有理式的不定积分

将 $\frac{P(x)}{Q(x)}$ 分解，出现 $\frac{A}{(x-\alpha)^r}$ 和 $\frac{Bx+C}{(x^2+\beta x+\gamma)^s}$。

$$\int \frac{A}{(x-\alpha)^r}dx = \begin{cases} A\ln|x-\alpha| + C, & r=1 \\ -\frac{A}{r-1}\cdot\frac{1}{u^{r-1}} + C, & r\neq 1 \end{cases}$$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^s}dx = \int \frac{(2x+\beta)\frac{B}{2} + C - \frac{\beta B}{2}}{(x^2+\beta x+\gamma)^s}dx$$

$$= \frac{B}{2}\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^s}dx + (C - \frac{\beta B}{2})\int \frac{1}{(x^2+\beta x+\gamma)^s}dx$$

$$= \frac{B}{2}\int \frac{du}{u^s} + (C - \frac{\beta B}{2})\int \frac{dx}{[(x+\frac{\beta}{2})^2 + \gamma - \frac{\beta^2}{4}]^s}$$

$$= \frac{B}{2}\int \frac{du}{u^s} + (C - \frac{\beta B}{2})\int \frac{dv}{(v^2+a^2)^s}$$

$$= \begin{cases} \frac{B}{2}\ln|u|, & s=1 \\ -\frac{B}{2}\cdot\frac{1}{s-1}\cdot\frac{1}{u^{s-1}}, & s\neq 1 \end{cases} + \begin{cases} (C - \frac{\beta B}{2})\frac{1}{a}\arctan\frac{v}{a} + C, & s=1 \\ \text{书例递推}, & s\neq 1 \end{cases}$$

$I_{n+1} = \frac{v}{2na^2(v^2+a^2)^n} + \frac{2n-1}{2na^2}I_n$
$I_1 = \frac{1}{a}\arctan\frac{x}{a}$。

例. $\int \frac{1}{x^3+1}dx$

$= \int \frac{1}{(x+1)(x^2-x+1)}dx = \int (\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1})dx$

$= \int (\frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1})dx$

$= \frac{1}{3}\ln|x+1| + \int \frac{-\frac{1}{6}(2x-1) + \frac{2}{3} - \frac{1}{6}}{(x-\frac{1}{2})^2 + \frac{3}{4}}dx$

$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\int \frac{d(x^2-x+1)}{x^2-x+1} + \int \frac{\frac{1}{2}}{x^2-x+1}dx$

$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int \frac{d(x-\frac{1}{2})}{(x-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$

$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$

$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

$= \frac{1}{6}\ln\frac{x^2+2x+1}{x^2-x+1} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$.

例. $I = \int \frac{1-x^6}{x(1+x^6)}dx$，$J = \int \frac{x^4+1}{x^6+1}dx$

$I = \int \frac{1-x^6}{x(1+x^6)}dx = \int \frac{1+x^6-2x^6}{x(1+x^6)}dx$

$= \int (\frac{1}{x} - \frac{2x^5}{(1+x^6)})dx$

$= \ln|x| - 2\int \frac{x^5}{1+x^6}dx = \ln|x| - \frac{1}{3}\int \frac{d(x^6+1)}{x^6+1}$

$= \ln|x| - \frac{1}{3}\ln(x^6+1) + C$.

$J = \int \frac{x^4+x^2-x^2+1}{(x^2+1)(x^4-x^2+1)}dx$

$= \int (\frac{x^2}{x^6+1} + \frac{x^4}{x^2+1})dx$

$= \int (\frac{x^2}{x^6+1} + \frac{x^4-x^2+1}{x^6+1})dx$

$= \int \frac{1}{x^2+1}dx + \frac{1}{3}\int \frac{d(x^3)}{1+(x^3)^2}$

$= \arctan x + \frac{1}{3}\arctan(x^3) + C$.

例. $\int \frac{1}{x^3+1}dx$

$= \int \frac{dx}{(x+1)(x^2-x+1)} \Rightarrow \frac{1}{x^3+1} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)}$，

待定系数 $1 = A(x^2-x+1) + (x+1)(Bx+C)$

[$(\ \cdots\ )\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ 行列式 $\neq 0$，必有解。]

$\therefore A = \frac{1}{3}$，$B = -\frac{1}{3}$，$C = \frac{2}{3}$。

$\therefore$ 原式 $= \ln|x+1| + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}$（换元）。

例. $\int \frac{1}{x(x+1)(x^2+x+1)}dx = \int \frac{x^2+x+1-x^2-x}{x(x+1)(x^2+x+1)}dx$

$= \int \frac{dx}{x(x+1)} - \int \frac{dx}{x^2+x+1} = \int \frac{dx}{x} - \int \frac{dx}{x+1} + \int \frac{dx}{x^2+x+1}$
（ln，ln，arctan）

例. $\int \frac{dx}{x(x^8+1)}$

$= \int \frac{1+x^8-x^8}{x(x^8+1)}dx = \int \frac{1}{x}dx - \int \frac{x^7}{x^8+1}dx$

$= \ln|x| - \frac{1}{8}\int \frac{dx^8}{x^8+1} = \ln|x| - \frac{1}{8}\ln(x^8+1) + C$.

四、三角有理式的不定积分

1. $R(u,v) = \frac{\sum_{i,j=0}^{n,m} a_{ij}u^iv^j}{\sum_{i,j=0}^{k,l} b_{ij}u^iv^j} \longrightarrow \int R(\sin x, \cos x)\,dx$.

万能变换，令 $t = \tan\frac{x}{2}$，$x = 2\arctan t$.
$t = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$，$t^2 = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}$，$\cos^2\frac{x}{2} = \frac{1}{1+t^2}$，$\sin^2\frac{x}{2} = \frac{t^2}{1+t^2}$
$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$，$\sin x = -\frac{2t}{1+t^2}$
$dx = \frac{2}{1+t^2}dt$.
则 $\int R(\sin x, \cos x)dx = \int R(\frac{1-t^2}{1+t^2}, -\frac{2t}{1+t^2})\frac{2}{1+t^2}dt$

例. $\int \frac{dx}{5+4\sin x} = \int \frac{\frac{2}{1+t^2}}{5+4\cdot\frac{-2t}{1+t^2}}dt = \int \frac{2}{5t^2+8t+5}dt$

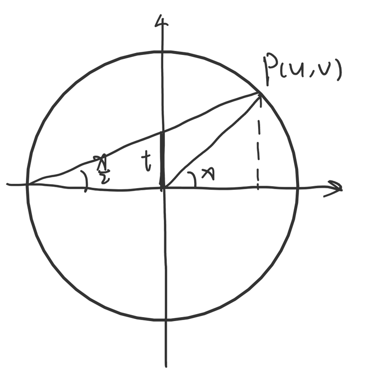
$= \frac{2}{5}\int \frac{dt}{(t+\frac{4}{5})^2 + 1 - \frac{16}{25}}$.

例. $\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx = \int (\frac{1+\cos 2x}{2})^2 dx$

$= \frac{1}{4}\int (1 - \cos 2x + \cos^2 2x)\,dx$

$= \frac{1}{4}(x + \sin x + \int \cos^2 2x\,dx)$

2. 几何意义



$t = \tan\frac{x}{2}$.
$u = \frac{1-t^2}{1+t^2}$，$v = \frac{2t}{1+t^2}$
$u^2 + v^2 = 1$
$(1-t^2)^2 + (2t)^2 = (1+t^2)^2$.
$x \to t \to u, v$.

五、其他换元方法.

1. 双曲函数 $\int R(\operatorname{sh} x, \operatorname{ch} x)\,dx$
   ↓ $u = \tanh x$
   u 的有理函数.

2. 根式

① $\int R(x, \sqrt{1-x^2})\,dx$
↓ $x = \cos u$，$\sqrt{1-x^2} = \sin u$ $\quad dx = -\sin u\,du$.
$\int R(\cos u, \sin u)(-\sin u)du$.
↓ $t = \tan\frac{u}{2}$
② → ①
$t = \frac{\sin\frac{u}{2}}{\cos\frac{u}{2}} = \sqrt{\frac{1-\cos u}{1+\cos u}} = \sqrt{\frac{1-x}{1+x}}$.

② $\int R(x, \sqrt{x^2-1})\,dx$.
↓ $x = \cosh u$，$\sqrt{x^2-1} = \sinh u$. $\quad dx = \sinh u\,du$.
$\int R(\cosh u, \sinh u)\sinh u\,du$ (③)
↓
$t = \frac{\operatorname{sh}\frac{u}{2}}{\operatorname{ch}\frac{u}{2}}$.

③ $\int R(x, \sqrt{x^2+1})\,dx$
↓ $x = \operatorname{sh} u$，$\sqrt{x^2+1} = \operatorname{ch} u$
$\int R(\operatorname{sh} u, \operatorname{ch} u)\cdot \operatorname{ch} u\,du$.

④ $\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}})\,dx$，换元 $t = \sqrt[n]{\frac{ax+b}{cx+d}}$.
$(cx+d)t^n = ax+b$，$x = \frac{b-dt^n}{ct^n-a} = \varphi(t)$
$\int R(\frac{b-dt^n}{ct^n-a}, t)\,d(\varphi(t))$
$= \int R(\varphi(t), t)\varphi'(t)\,dt$.

⑤ $\int R(x, \sqrt{x^2+bx+c})\,dx$ 作 Euler 代换
$\sqrt{x^2+bx+c} = t \pm x$，$x^2+bx+c = x^2 + 2tx + t^2$，
$x = \frac{c-t^2}{2t-b} = \varphi(t)$.
$\int R(\varphi(t), t + \varphi(t))\,d(\varphi(t))$

例. 计算 $I = \int \frac{1}{5+4\sin x}dx$，
$J_1 = \int \frac{1}{(\cos x+\sin x)^2}dx$. $J_2 = \int \frac{1}{\sin^4 x\cos^2 x}dx$

(1) $I = \int \frac{dx}{5+4\sin x} \overset{t=\tan\frac{x}{2}}{=\!=} \int \frac{d(2\arctan t)}{5+4\times\frac{2t}{1+t^2}}$

$= \int \frac{\frac{2}{1+t^2}}{5+\frac{8t}{1+t^2}}dt = \int \frac{2}{5t^2+8t+5}dt$ → $\Delta < 0$

$= \int \frac{2dt}{5((t+\frac{4}{5})^2 + (\frac{3}{5})^2)} = \frac{2}{5}\int \frac{d(\frac{4}{5}+t)}{(\frac{4}{5}+t)^2 + (\frac{3}{5})^2}$

$= \frac{2}{5}\int \frac{du}{u^2+(\frac{3}{5})^2} = \frac{2}{5}\times\frac{5}{3}\arctan\frac{u}{\frac{3}{5}}$

$= \frac{2}{3}\arctan\frac{t+\frac{4}{5}}{\frac{3}{5}} = \frac{2}{3}\arctan\frac{5t+4}{3} + C$.

$= \frac{2}{3}\arctan\frac{5\tan\frac{x}{2}+4}{3} + C$.

(2) $\int \frac{1}{(\cos x+\sin x)^2}dx = \int \frac{1}{1+2\sin x\cos x}dx$

$= \int \frac{1}{1+\sin 2x}dx =$

$= \int \frac{dx}{(1+2\tan x+\tan^2 x)\cos^2 x} = \int \frac{d(1+\tan x)}{(1+\tan x)^2}$

$= -\frac{1}{1+\tan x} + C$

(3) $\int \frac{1}{\sin^4 x\cos^2 x}dx$

$= \int \frac{(\cos^2 x+\sin^2 x)^2}{\sin^2 x\sin^2 x(1-\sin^2 x)}dx$ 分子低两次.

$= \int \frac{(1+\tan^2 x)^2}{\tan^4 x}\frac{\cos^4 x}{\cos^6 x}dx$ $\Rightarrow \int R(\tan x)\frac{\cos^{n-2}x}{\cos^n x}dx = \int R(\tan x)\,d\tan x$.

$= \int \frac{(1+\tan^2 x)^2}{\tan^4 x}d\tan x$

$= \int (\frac{1}{t^4} + \frac{2}{t^2} + 1)\,dt$

例. $\int \sqrt{\frac{2-x}{x+1}}dx \quad (-1 < x < 2)$ $\qquad t^2 - 1 - 2 + t^2$

令 $t = \sqrt{\frac{2-x}{x+1}}$，$t^2 = \frac{2-x}{x+1}$，$x = \frac{2-t^2}{t^2-1} = \frac{3}{t^2+1} - 1$.

$\int t\,d(\frac{3}{t^2+1} - 1) = \int t\times\frac{-6t}{(t^2+1)^2}dt$

$= -6\int \frac{t^2}{(t^2+1)^2}dt = -6\int \frac{(t^2+1)-1}{(t^2+1)^2}dt$

$= -6\int \frac{dt}{(t^2+1)} + 6\int \frac{dt}{(t^2+1)^2}$

例. 求 $\int \frac{1}{x\sqrt{x^2+x-1}}dx \quad (x > \frac{-1+\sqrt{5}}{2},\ x < \frac{-1-\sqrt{5}}{2})$.

令 $\sqrt{x^2+x-1} = t - x$，$x^2+x-1 = x^2 - 2tx + t^2$，

$x = \frac{t^2+1}{2t+1}$. $\int \frac{1}{\frac{t^2+1}{2t+1}(t - \frac{t^2+1}{2t+1})}d\frac{t^2+1}{2t+1}$

$= \int \frac{2t+1}{t^2+1}\times\frac{2t+1}{t^2+t-1}\times\frac{2t(2t+1) - (t^2+1)2}{(2t+1)^2}dt$

$= \int \frac{4t^2+2t-2t^2-2}{(t^2+1)(t^2+t-1)}dt = \int \frac{2}{t^2+1}dt$

$= 2\arctan t + C = 2\arctan(x + \sqrt{x^2+x-1}) + C$

原式 $= \int \frac{1}{x^2\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}}dx$

$= -\int \frac{d(\frac{1}{x})}{\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}} = -\int \frac{d(\frac{1}{x})}{\sqrt{\frac{5}{4} - (\frac{1}{2}-\frac{1}{x})^2}}$

$= \int \frac{d(\frac{1}{2}-\frac{1}{x})}{\sqrt{\frac{5}{4} - (\frac{1}{2}-\frac{1}{x})^2}} = \int \frac{du}{\sqrt{\frac{5}{4} - u^2}}$

$= \frac{4}{5}\arcsin\frac{2}{\sqrt{5}}u + C$

一、定义.

假分式：分子次数 ≥ 分母次数
真分式：分子次数严格小于分母.
假分式 $\xrightarrow{\text{辗转相除法}}$ 多项式 + 真分式.
称形如 $\frac{A}{(x-\alpha)^k}$，$\frac{Bx+C}{(x^2+\beta x+\gamma)^k}$ 的分式为简单分式.
真分式可以被分为若干简单分式之和.

例. 写出下列真分式的简单分式分拆.
$r_1(x) = \frac{P(x)}{(x-\alpha)^3}$. $\quad r_2(x) = \frac{P(x)}{(x^2+\beta x+\gamma)^2}$

$r_1(x) = \frac{A_1}{(x-\alpha)} + \frac{A_2}{(x-\alpha)^2} + \frac{A_3}{(x-\alpha)^3}$

$r_2(x) = \frac{B_1x+C_1}{x^2+\beta x+\gamma} + \frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2}$

$r_3(x) = \frac{P(x)}{(x-\alpha)^3(x^2+\beta x+\gamma)^2}$

$= \frac{A_{11}}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \frac{A_3}{(x-\alpha)^3} + \frac{B_1x+C_1}{x^2+\beta x+\gamma} + \frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2}$.

$A_3 = \frac{P(x)}{(x^2+\beta x+\gamma)^2}\Big|_{x=\alpha}$.

例. $\int \frac{1}{x^2-a^2}dx$

$= \int \frac{1}{(x+a)(x-a)}dx = \frac{1}{2a}\int (\frac{1}{x-a} - \frac{1}{x+a})dx$

$= \frac{1}{2a}\ln|\frac{1}{x-a}| - \frac{1}{2a}\ln|\frac{1}{x+a}| + C$.

例. $\int \frac{x^2}{(x-1)^3}dx$

$\to \int \frac{(t+1)^2}{t^3}dt = \int (\frac{1}{t} + \frac{2}{t^2} + \frac{1}{t^3})dt$

$= \ln|t| - 2\frac{1}{t} - \frac{1}{2}\frac{1}{t^2} + C$

$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C$

转化为有理式积分：

① 三角有理式. $\int R(\cos x, \sin x) = \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}$
作万能代换 $t = \tan\frac{x}{2}$，$x = 2\arctan t$，
$\int R(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})\,d\arctan\frac{t}{2}$
$= \int R(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})\frac{2}{1+t^2}dt$.

② $\int R(\tan\alpha x)\,dx \overset{t=\tan\alpha x}{\underset{x=\frac{1}{\alpha}\arctan t}{=\!=}} \int R(t)\frac{1}{\alpha}\cdot\frac{1}{1+t^2}dt$.

③ $\int R(\sin x)\cos x\,dx$，$\int R(\cos x)\sin x\,dx$，凑微分.
如 $\int \sin^3 x\,dx = -\int \sin^2 x\,d\cos x = -\int (1-\cos^2 x)\,d\cos x$
$\int e^{\sin x}\cos x = \int e^{\sin x}\,d\sin x = e^{\sin x} + C$.