证明の重开

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一、导数性质

1. 四则运算

$$(f(x)+g(x))' = \lim_{\Delta x\to 0}\frac{(f(x+\Delta x)+g(x+\Delta x))-(f(x)+g(x))}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}$$
$$= f'(x)+g'(x).$$

$$(f(x)g(x))' = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x+\Delta x)+f(x)g(x+\Delta x)-f(x)g(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(f(x+\Delta x)-f(x))}{\Delta x}g(x+\Delta x)+f(x)\frac{g(x+\Delta x)-g(x)}{\Delta x}$$
$$= f'(x)g(x)+f(x)g'(x).$$

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f(x)\left(\frac{1}{g(x)}\right)' + f'(x)\cdot\frac{1}{g(x)}$$

$$\left(\frac{1}{g(x)}\right)' = \lim_{\Delta x\to 0}\frac{\frac{1}{g(x+\Delta x)}-\frac{1}{g(x)}}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{1}{g(x+\Delta x)g(x)}\times\frac{g(x)-g(x+\Delta x)}{\Delta x}$$
$$= \frac{-g'(x)}{g(x)^2}$$

2. 复合函数求导.

3. 反函数求导.

$y=f(x)$, $x=f^{-1}(y)$.

$$(f^{-1}(y))' = \lim_{y_1\to y_2}\frac{f^{-1}(y_1)-f^{-1}(y_2)}{y_1-y_2} = \lim_{x_1\to x_2}\frac{x_1-x_2}{f(x_1)-f(x_2)} = \frac{1}{f'(x)}$$

4. 参数方程表示的函数的导数.

$x=\varphi(t)$, $y=\psi(t)$.

$t=\varphi^{-1}(x)$, $y=\psi(t)=\psi(\varphi^{-1}(x))$

$$\frac{dy}{dx} = \psi'(\varphi^{-1}(x))\cdot(\varphi^{-1}(x))' = \psi'(\varphi^{-1}(x))\frac{1}{\varphi'(t)} = \frac{\psi'(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))}.$$

$$\frac{d^2y}{dx^2} = \left(\frac{\psi'(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))}\right)' = \frac{[\psi'(\varphi^{-1}(x))]'\varphi'(\varphi^{-1}(x)) - \psi'(\varphi^{-1}(x))[\varphi'(\varphi^{-1}(x))]'}{[\varphi'(\varphi^{-1}(x))]^2}$$

$$= \frac{\psi''(\varphi^{-1}(x))[\varphi^{-1}(x)]'\varphi'(\varphi^{-1}(x)) - \psi'(\varphi^{-1}(x))\varphi''[\varphi^{-1}(x)][\varphi^{-1}(x)]'}{[\varphi'(\varphi^{-1}(x))]^2}$$

$\varphi'(\varphi(x))$

$$= \frac{\psi''(\varphi^{-1}(x)) - \psi'(\varphi^{-1}(x))\varphi''(\varphi^{-1}(x))}{(\varphi'(\varphi^{-1}(x)))^3}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{\psi'(t)}{\varphi'(t)}\right)(\varphi^{-1}(x))'$$

2.0
$$\begin{cases} x=\varphi(t) \\ y=\psi(t). \end{cases} \Rightarrow \begin{aligned} &t=\varphi^{-1}(x). \\ &y=\psi(t)=\psi(\varphi^{-1}(x)). \end{aligned}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \psi'(t)\times\frac{1}{\psi'(\varphi^{-1}(x))} = \frac{\psi'(\varphi(x))}{\psi'(\varphi^{-1}(x))}$$
$$= \psi'(t)\times\frac{1}{\varphi^{-1}(t)} = \frac{\psi'(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \psi'(t)\times\frac{1}{\varphi'(t)} = \frac{\psi'(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\times\frac{\psi'(t)}{\varphi'(t)}\times\frac{d(t)}{dx}$$
$$= \frac{\psi''(t)\varphi'(t)-\psi'(t)\varphi''(t)}{[\varphi'(t)]^2}$$
$$= \frac{\psi''(\varphi^{-1}(x))\varphi'(\varphi'}{[\varphi'(\varphi^{-1}(x))]}$$

二、中值定理.

$f(x)$ 在 $[a,b]$ 连续，(a,b) 可导.

1. Fermat：$f(x_0)$ 为极值，$f'(x_0)=0$

证明：不妨设 $f(x_0)$ 为极大值.

$x\in[a,x_0]$，$f(x)\le f(x_0)$. $x\in[x_0,b]$，$f(x)\le f(x_0)$.

$$f'_+(x_0)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\le 0$$
$$f'_-(x_0)=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\ge 0 \quad\Rightarrow f'(x_0)=0.$$

2. Rolle：$f(a)=f(b)$，$\exists\xi\in[a,b]$，$f'(\xi)=0$.

证明：连续 $\Rightarrow$ 有最大值 $f(x_1)$/最小值 $f(x_2)$

i) 最值在端点取. $f(a)=f(b)=f(x_1)=f(x_2)$.

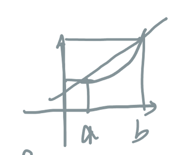
$\therefore f(x)=C$，$f'(x)=0$

ii) 至少有一个最值不在端点.

则有 $f(x_1)$. $f(x_1)$ 是 $[a,b]$ 上的极大/小值.

$\therefore f(x_1)=f'(x_1)=0$

3. Lagrange：$\exists\xi\in[a,b]$，$f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$.



证明：构造 $F(x)=\dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)-f(x)$

$F(a)=0$，$F(b)=0$.

根据 Rolle，$\exists\xi\in[a,b]$，$F'(\xi)=0$

$\therefore \dfrac{f(b)-f(a)}{b-a}-f'(\xi)=0$

4. Cauchy：$\exists\xi[a,b]$，$\dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(b)-f(a)}{g(a)-g(b)}$.

证明：构造 $F(x)=\dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))+f(a)-f(x)$.

$F(a)=0$，$F(b)=0$　同上.

5. 导函数的极限：$f(x)$ 在 I 上连续可导.

若在 x_0 处导数右极限 $f'(x_0+0)=\lim_{x\to x_0^+}f'(x)$ 存在，则 x_0 处右导数 $f'_+(x_0)=f'(x_0+0)$.

证明：
$$\left[\begin{aligned}&\text{右极限 } f'(x_0+0)=\lim_{x\to x_0^+}f'(x).\\ &\text{右导数 } f'_+(x_0)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\end{aligned}\right]$$

对 $\forall x>x_0$，$\exists\xi\in(x,x_0)$，使 $f'(\xi)=\dfrac{f(x)-f(x_0)}{x-x_0}$.

$x\to x_0^+$，$\lim_{x\to x_0^+}f'(\xi)=\lim_{x\to x_0^+}\dfrac{f(x)-f(x_0)}{x-x_0}=f'_+(x_0)$.

又 $\lim_{x\to x_0^+}f'(\xi)=\lim_{x\to x_0^+}f'(x)=f'(x_0+0)$

$\therefore$ 得证.

6. 导函数的介值性：$f(x)$ 在 I 上连续可导，则对 $\forall\lambda$ 满足 $f'_+(a)<\lambda<f'_-(b)$，都 $\exists\xi\in(a,b)$，使 $f'(\xi)=\lambda$.

证明：设 $F(x)=f(x)-\lambda x$，$F'(x)=f'(x)-\lambda$.

$F'_+(a)=f'_+(a)-\lambda<0$，$F'_-(b)=f'_-(b)-\lambda>0$，

对 $\forall\delta$，$\exists x_1\in(a,a+\delta)$，使 $F'(x_1)=\dfrac{F(a+\delta)-F(a)}{\delta}<0$

$\therefore F(a+\delta)<F(a)$，即 $F(a)$ 不是最小值.

同理，$F(b)$ 不是最小值

又 $\because F(x)$ 在 (a,b) 连续可导，

$\therefore F(x)$ 在 (a,b) 上有最小值 $F(\xi)$，$\xi\in(a,b)$.

$\therefore F'(\xi)=0$，$f'(\xi)=\lambda$.

三、L'hospital.

四. 导数与函数.

$\lim_{x\to x_0^+}f'(x)=f'(x_0+0)$ 存在

$\Downarrow$

$f'_+(x_0)$ 存在 $=f'(x_0+0)$

$\lim_{x\to x_0^+}f(x)=$

五. Taylor 展开.

1. Taylor 公式.

$f(x)$ 有 n 阶导数，则 $f(x)=T_n(x)+R(x)$，

若 $T_n(x)$ 为多项式 $a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n$，

则有各系数 $a_k=\dfrac{f^{(k)}(x)}{k!}$，$k=1,2\cdots$

证明：$x=x_0$ 时，$f(x)=f(x_0)=a_0$.

假设 $n=k$ 时，$a_k=\dfrac{f^{(k)}(x)}{k!}$ 成立.

$$\frac{f(x)-T_k(x)}{(x-x_0)^{k+1}}=\frac{o((x-x_0)^{k+1})}{(x-x_0)^{k+1}}$$
$$\lim_{x\to x_0}\frac{f(x)-T_n(x)}{(x-x_0)^{k+1}}=\lim_{x\to x_0}\frac{f'(x)-T'_n(x)}{(k+1)(x-x_0)^k}=\cdots$$
$$=\lim_{x\to x_0}\frac{f^{(k+1)}(x)-T_n^{(k+1)}(x)}{(k+1)!(x-x_0)}=f^{(k+1)}(x)-(k+1)!\,a_{k+1}=0.$$

2. 带 Peano 余项的 Taylor 公式.

$f(x)=T_n(x)+o((x-x_0)^n)$.

证明：（只需证 $R_n(x)\sim(x-x_0^n)$）.

$$\lim_{x\to x_0}\frac{f(x)-T_n(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{f'(x)-T_n'(x)}{n(x-x_0)^{n-1}}=\cdots$$
$$=\lim_{x\to x_0}\frac{f^{(n-1)}(x)-T_n^{(n-1)}(x)}{n!(x-x_0)}=\lim_{x\to x_0}\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)}$$
$$=\left[\lim_{x\to x_0}\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)}{x-x_0}-\lim_{x\to x_0}f^{(n)}(x_0)\right]\frac{1}{n!}=0.$$

3. 带 Lagrange 余项的 Taylor 公式.

$f(x)=T_n(x)+\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$　$(n+1)$ 阶可导).

证明：只需证 $R_n=f(x)-T_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

设 $g(t)=R_n(t)-\dfrac{R_n(x)}{(x-x_0)^{n+1}}(t-x_0)^{n+1}$.

$g(x)=R_n(x)-\dfrac{R_n(x)}{(x-x_0)^{n+1}}(x-x_0)^{n+1}=0$，

$g(x_0)=R_n(x_0)-\dfrac{R_n(x)}{(x-x_0)^{n+1}}(x_0-x_0)^{n+1}$

$=R_n(x_0)=f(x_0)-[f(x_0)+f'(x_0)(x_0-x_0)+\cdots]=0$

$g'(x_0)=R_n'(x_0)-\dfrac{R_n(x)}{(x-x_0)^{n+1}}(n+1)(x_0-x_0)^n$

$=f'(x_0)-[f'(x_0)(x_0-x_0)+f''(x_0)(x_0-x_0)^2+\cdots]=0$

$g''(x_0)=\cdots g^{(n)}(x_0)=0$

$\exists\xi_0\in(x,x_0)$，$g'(\xi_0)=0$

$\exists\xi_1\in(x,\xi_0)$，$g''(\xi_1)=0$

…

$\exists\xi_{n-1}\in(x,\xi_{n-2})$，$g^{(n)}(\xi_{n-1})=0$

$\exists\xi\in(x,\xi_{n-1})$，$g^{(n+1)}(\xi)=0$.

$g^{(n+1)}(\xi)=R_n^{(n+1)}(\xi)-\dfrac{R_n(x)}{(x-x_0)^{n+1}}(n+1)!$

$=f^{(n+1)}(\xi)-\dfrac{R_n(x)}{(x-x_0)^{n+1}}(n+1)!=0$.

$R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$.