补充题

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习题指导

1. (1) $\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0-\Delta x)+f(x_0-\Delta x)-f(x_0)}{\Delta x}$ (加减同一项)

$=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x_0-\Delta x)-f(x_0)}{\Delta x}$ (极限的和)

$=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}\cdot 2+\lim_{\Delta x\to 0}\frac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}\cdot(-1)$

$=2f'(x_0)+f'(x_0)=3f'(x_0)$.

(2) $\lim_{\Delta x\to 0}\frac{x_0^2-(x_0-\Delta x)^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x_0\Delta x-\Delta x^2}{\Delta x}=\lim_{\Delta x\to 0}(2x_0-\Delta x)=2x_0$.

(3) $\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)+f(x_0)-f(x_0-\Delta x)}{\Delta x}$

$=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}=f'(x_0)+f'(x_0)=2f'(x_0)$.

2. $f(x+y)=f(x)+f(y)$

$f(y)=f(x+y)-f(x)$. $\lim_{y\to 0}\frac{f(x+y)-f(x)}{y}=\lim_{y\to 0}\frac{f(y)}{y}=f'(0)$.

令 $x+y=0$，$\lim_{x\to 0}f(x)=0$.

$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\Delta x)-f(0)}{\Delta x}=f'(x)$.

3. $f(1+\sin x)=3f(1-\sin x)+8x+o(x)\quad(x\to 0)$

$\forall\varepsilon>0$，$\exists\delta(\varepsilon,\delta)$，$\forall x$，$f(x)$ 可微 $\Rightarrow f(x)$ 在 $(1,f(1))$ 处的切线为 $y=f(1)+f'(1)(x-1)$.

令 $f(1)=a$，$f'(1)=b$，$\lim_{x\to 0}\frac{f(1+\sin x)-3f(1-\sin x)-8x}{x}=0$

$\lim_{x\to 0}\frac{f(1+\sin x)-3f(1-\sin x)-8x}{\sin x}=\lim_{x\to 0}\frac{a+b\sin x-3a+3b\sin x-8x}{\sin x}=0$

$\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}+3\lim_{x\to 0}\frac{f(1-\sin x)-f(1)}{-\sin x}=8$

$\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}=\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{x}\cdot\frac{x}{\sin x}=f'(1)\cdot 1=f'(1)=b$

$\lim_{x\to 0}\frac{f(1-\sin x)-f(1)}{-\sin x}=\lim_{x\to 0}\frac{f(1-\sin x)-f(1)}{-x}\cdot\frac{-x}{-\sin x}=f'(1)\cdot 1=f'(1)=b$

$\therefore b+3b=8$，$b=2$.

$\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}=\lim_{x\to 0}\frac{2\sin x+o(\sin x)}{\sin x}=2+\lim_{x\to 0}\frac{o(\sin x)}{\sin x}=2$

$\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{x}=\lim_{x\to 0}\frac{f(1+\sin x)-f(1)}{\sin x}\cdot\frac{\sin x}{x}=2\cdot 1=2$

$\therefore f'(1)=2$

4. $y=f\left(\frac{3x-2}{3x+2}\right)$，$f(x)=\arcsin x^2$，$\left.\frac{dy}{dx}\right|_{x=0}$

$\frac{dy}{dx}=\frac{df\left(\frac{3x-2}{3x+2}\right)}{dx}=\frac{df\left(\frac{3x-2}{3x+2}\right)}{d\frac{3x-2}{3x+2}}\cdot\frac{d\frac{3x-2}{3x+2}}{dx}$ (链式法则)

$=f'\left(\frac{3x-2}{3x+2}\right)\cdot\left(\frac{3x-2}{3x+2}\right)'$，$f'(x)=\frac{2x}{\sqrt{1-x^4}}$，$\left(\frac{3x-2}{3x+2}\right)'=\frac{12}{(3x+2)^2}$

$\therefore\left.\frac{dy}{dx}\right|_{x=0}=f'(-2)\cdot 3=\frac{4}{\sqrt{15}}\cdot 3=\frac{12}{\sqrt{15}}=\frac{4\sqrt{15}}{5}$

5. x^3+y^3-

$y\ln x-x\ln y=0$：$\frac{dy}{dx}=\frac{\ln y}{\frac{x}{y}}$，$y\ln x-x\ln y=0$

$f(x)\ln x-x\ln f(x)=0$：$f'(x)\ln x+\frac{f(x)}{x}-\ln f(x)-x\frac{f'(x)}{f(x)}=0$

$f(x)(\ln x-\frac{x}{f(x)})=f'(x)(\ln x-\frac{x}{f(x)})$

$\frac{dy}{dx}=\frac{\ln y-\frac{y}{x}}{\ln x-\frac{x}{y}}=\frac{x\ln y-y^2}{x\ln x-xy}=\frac{y'(x\ln x-y)}{x(\ln x-\ln y)}$

6. $y=y(x)$：$\begin{cases}x=e^t+2t+1\\ e^y\sin t-y+1=0\end{cases}$，$\left.\frac{dy}{dx}\right|_{t=0}$，$\left.\frac{d^2y}{dx^2}\right|_{t=0}$

$x=\varphi(t)$，$y=\psi(t)$，$t=\varphi^{-1}(x)$.

$\frac{dy}{dx}=\frac{d\,\psi(\varphi^{-1}(x))}{dx}=\psi'(\varphi^{-1}(x))\cdot(\varphi^{-1}(x))'$

$\frac{dy}{dt}=\psi'(t)$，$\frac{dx}{dt}=\varphi'(t)$，$e^{f(t)}\sin t-f(t)+1=0$，$f(0)=2$，$\frac{df}{dt}=\frac{-e^{f(t)}\sin t}{e^{f(t)}\cos t-1}$

$\frac{dy}{dx}=\frac{e^{f(t)}\sin t}{1-e^{f(t)}\cos t}$，$\left.\frac{dy}{dx}\right|_{t=0}=\frac{0}{2-1}=0$，$\frac{dx}{dt}=e^t+2$

$\frac{d^2y}{dx^2}=\frac{d\frac{dy}{dx}}{dx}=\frac{d\frac{dy}{dx}}{dt}\cdot\frac{dt}{dx}=\frac{e^{f(t)}\sin t}{2-e^t}\cdot\frac{1}{e^t+2}$

$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}=\frac{e^{f(t)}\sin t}{(2-e^t)(e^t+2)}\cdot\frac{1}{e^t+2}$

$=\frac{e^{f(t)}\cos t\cdot f'(t)\cdot(2-e^t)-e^{f(t)}\sin t\cdot(-e^t)}{(2-e^t)^2(e^t+2)^2}$，$f'(t)=\frac{e^{f(t)}\sin t}{e^{f(t)}\cos t-1}$

$=\frac{e^{f(t)}\sin t\cdot(2-e^t)\cdot\cos t+e^{f(t)}\sin t\cdot e^t(e^{f(t)}\cos t-1)}{(2-e^t)^2(e^t+2)^2(e^{f(t)}\cos t-1)}$

$t=0$，$f(0)=2$ 代入，$\left.\frac{d^2y}{dx^2}\right|_{t=0}=\frac{0+0}{1\cdot 3\cdot 3\cdot(2-1)}=0$

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7. $(1+\frac{1}{x})^{x+1}<e^x$.

$x\ln(1+\frac{1}{x})<x$，$\ln(1+\frac{1}{x})<\frac{1}{x}$，$\frac{\ln(1+\frac{1}{x})-\ln 1}{(1+\frac{1}{x})-1}<1$.

$f(x)=\ln^2 x$，$f'(x)=\frac{2}{x}\downarrow$.

$\exists\xi\in(1,\frac{n}{n-1})$，$f'(\xi)=\frac{f(\frac{n}{n-1})-f(1)}{\frac{n}{n-1}-1}=\frac{1}{\xi}<1$，$(\frac{n}{n-1}>1,x=n)$

$\exists\eta\in(\frac{n}{n-1},1)$，$f'(\eta)=\frac{f(\frac{n}{n-1})-f(1)}{\frac{n}{n-1}-1}=\frac{2}{\eta}$，$(\frac{n}{n-1}<1)$

8. $f(x)$ 在 (a,b) 内可导，$\forall x_0\in(a,b)$，$f'(x_0)\neq 0$.

$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\neq 0$.

$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}<0$

$\exists\delta>0$，$\forall x\in(x_0-\delta,x_0+\delta)$，$\frac{f(x)-f(x_0)}{x-x_0}<0$.

$\forall x\in(x_0-\delta,x_0)$，$f(x)>f(x_0)$；$\forall x\in(x_0,x_0+\delta)$，$f(x)<f(x_0)$.

$\forall x_1\in(x_0-\delta,x_0)$，$\exists x_2\in(x_0,x_0+\delta)$，$f(x_1)=f(x_2)$.

$\forall x_1,x_2\in(a,b)$，$x_1<x_2$，$\exists x_0\in(x_1,x_2)$，$f(x_1)=f(x_2)$.

9. $f(x)$ 在 $\mathbb{R}$ 可导，则 $f(x)$ 在 $\mathbb{R}$ 上单调 $\Leftrightarrow$ 要证明两个方向.

(必要性) 设 $f(x)$ 在 x_0 处取到最值，不妨设为最小值.

$f(x_0+\Delta x)\geq f(x_0)$.

$\forall\Delta x\in(x_0-x_0,b-x_0)$：$\exists x_1$ 使 $f(x_1)=\frac{f(x_0+\Delta x)+f(x_0)}{2}$.

$\forall\Delta x\in(a-x_0,x_0-x_0)$：$\exists x_2$ 使 $f(x_2)=\frac{f(x_0+\Delta x)+f(x_0)}{2}$.

$f(x_0)=\lim_{\Delta x\to 0^+}f(x_1)=\lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)+f(x_0)}{2}=f(x_0)+\lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{2}$.

$f(x_0)=\lim_{\Delta x\to 0^-}f(x_2)=\lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)+f(x_0)}{2}=f(x_0)+\lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{2}$.

$\therefore\lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0)=0$.

12. $\lim_{x\to x_0}f(x)=\lim_{x\to x_0}f(x)=L$.

$\lim_{x\to x_0}f(x)=L$，对 $\forall\varepsilon>0$，$\exists\delta$，当 $0<x-x_0<\delta$ 时，

$|f(x)-L|<\frac{\varepsilon}{2}$，$L-\frac{\varepsilon}{2}<f(x)<L+\frac{\varepsilon}{2}$.

$\lim_{x\to x_0}f(x)=L$，对 $\forall\varepsilon>0$，$\exists X$，当 $x>X$ 时，

$|f(x)-L|<\frac{\varepsilon}{2}$，$L-\frac{\varepsilon}{2}<f(x)<L+\frac{\varepsilon}{2}$.

11. $\lim_{x\to 0}f(x)=0$，$f(x)=1$，$f(x+y)=e^x\cdot e^y$.

$f(x)=e^{x\cdot 0}\cdot f(0)=e^{x\cdot 0}\cdot 1$，$f(x)=e^{x\cdot 0}$.

$f(x)=e^x$，$\lim_{x\to 0}g(x)=\lim_{x\to 0}\frac{e^x-1}{x}=1$.

$\lim_{x\to 0}g(x)=\lim_{x\to 0}\frac{f(x)-1}{x}=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=1$，$f(0)=1$，$\exists\delta>0$，使 $f(x)>0$.

$\forall x_0\in(-\delta,\delta)$，$\lim_{x\to x_0}f(x)=\lim_{x\to x_0}f(x-x_0+x_0)=\lim_{x\to x_0}f(x-x_0)\cdot f(x_0)=f(x_0)\lim_{x\to x_0}f(x-x_0)$.

$\lim_{x\to x_0}f(x)=f(x_0)\lim_{x\to 0}f(x)=f(x_0)\lim_{x\to 0}\frac{f(x)-1}{x}\cdot x+f(x_0)=f(x_0)$.

$\therefore f(x)$ 在 $\mathbb{R}$ 上连续.

13. $f(x)$ 在 $[a,b]$ 上连续，$f(a)=a$，$f(b)=b$，$\exists\xi,\eta\in(a,b)$ 使 $f'(\xi)\cdot f'(\eta)=1$.

(证明) 令 $F(x)=f(x)-x$，$F(a)=0$，$F(\frac{a+b}{2})=f(\frac{a+b}{2})-\frac{a+b}{2}$，$F(b)=b-b=0$.

$\exists\xi\in(a,\frac{a+b}{2})$，使 $F(\xi)=0$.

$\exists\eta\in(\frac{a+b}{2},b)$，使 $F(\eta)=0$.

$g(x)=F(x)e^{kx}$，$g'(x)=e^{kx}(F'(x)-kF(x))=e^{kx}(f'(x)-1-k(f(x)-x))$.

$g(\xi)=F(\xi)e^{k\xi}$，$g'(\xi)=e^{k\xi}(F'(\xi)-kF(\xi))=e^{k\xi}(f'(\xi)-1-k(f(\xi)-\xi))=e^{k\xi}(f'(\xi)-1-k(\xi-\xi))=e^{k\xi}(f'(\xi)-1)$.

$g(\eta)=F(\eta)e^{k\eta}$，$g'(\eta)=e^{k\eta}(F'(\eta)-kF(\eta))=e^{k\eta}(f'(\eta)-1-k(f(\eta)-\eta))=e^{k\eta}(f'(\eta)-1)$.

$g(\xi)=g(\eta)=0$，$\exists k$，使 $f'(\xi)-1=0$，$f'(\eta)-1=0$.

$f'(\xi)\cdot f'(\eta)=1$.

14. $f(x)=\ln^2 x$，$\exists\xi\in(1,e)$，使 $f'(\xi)=f(e)$.

$f(x)=\ln^2 x$，$f(e)=1$，$f'(x)=\frac{2\ln x}{x}$，$f'(\xi)=\frac{2\ln\xi}{\xi}=1$，$\xi=2\ln\xi$.

$g(x)=x-2\ln x$，$g(1)=1>0$，$g(e)=e-2<0$，$\exists\xi\in(1,e)$，使 $g(\xi)=0$.

15. $f(x)$ 在 $[a,b]$ 上连续，$f(a)=f(b)=0$，$f(c)=1$，$c\in(a,b)$.

$\exists\xi\in(a,b)$，使 $f'(\xi)=2$.

$f(x)=\begin{cases}\frac{x}{a}f(a)+\frac{a-x}{a}f(c), & x\in[0,a]\\ \frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(c), & x\in[a,b]\end{cases}$

$f(x)$ 在 $[0,a]$，$[a,b]$ 上满足拉格朗日中值定理.

$\exists\xi_1\in(0,a)$，使 $f'(\xi_1)=\frac{f(a)-f(0)}{a-0}=\frac{1-0}{a}=\frac{1}{a}$.

$\exists\xi_2\in(a,b)$，使 $f'(\xi_2)=\frac{f(b)-f(a)}{b-a}=\frac{0-1}{b-a}=\frac{-1}{b-a}$.

$\because f'(\xi_1)\cdot f'(\xi_2)=-1$.

$\therefore$ 存在 $\xi=\xi_1$ 或 ξ_2，使 $f'(\xi)=2$.

16. $\lim_{x\to 0}\frac{e^{2x}-e^{3x}}{\ln(1+x)-x}$

$=\lim_{x\to 0}\frac{e^{2x}-e^{3x}}{x(\ln(1+x)-x)}=\lim_{x\to 0}\frac{e^{3x}(e^{-x}-1)}{x^2(\ln(1+x)-x)}$

$=\lim_{x\to 0}\frac{e^{3x}}{1}\cdot\frac{e^{-x}-1}{-x}\cdot\frac{-x}{\ln(1+x)-x}=\lim_{x\to 0}e^{3x}\cdot\frac{-x}{\ln(1+x)-x}$

$=\lim_{x\to 0}\frac{-x}{\ln(1+x)-x}=\lim_{x\to 0}\frac{-1}{\frac{1}{1+x}-1}=\lim_{x\to 0}\frac{-1}{-\frac{x}{1+x}}$

$=\lim_{x\to 0}\frac{1+x}{x}=\lim_{x\to 0}\left(\frac{1}{x}+1\right)=+\infty$

$\lim_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}=\lim_{x\to 0}\frac{\ln(1+\cos ax-1)}{\ln(1+\cos bx-1)}=\lim_{x\to 0}\frac{\cos ax-1}{\cos bx-1}$

$=\lim_{x\to 0}\frac{-\frac{1}{2}a^2x^2}{-\frac{1}{2}b^2x^2}=\frac{a^2}{b^2}$

$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}=\lim_{x\to 0}\frac{-x}{2x(1+x)}=-\frac{1}{2}$

$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{3x^2}=\lim_{x\to 0}\frac{-x}{3x^2(1+x)}=-\frac{1}{3}$

$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^n}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{nx^{n-1}}=\lim_{x\to 0}\frac{-x}{nx^{n-1}(1+x)}=\lim_{x\to 0}\frac{-1}{n(n-1)x^{n-2}(1+x)}=\begin{cases}-\frac{1}{n}, & n=1\\ \infty, & n>1\end{cases}$

17. $\lim_{x\to 0}f(x)=0$，$\lim_{x\to 0}\frac{f(x)}{x}=A$，$\lim_{x\to 0}f(x)=0$.

$\lim_{x\to 0}f(x)=0$，对 $\forall\varepsilon>0$，$\exists X_1>0$，$\forall x>X_1$，$|f(x)-0|<\frac{\varepsilon}{2}$.

$\lim_{x\to 0}\frac{f(x)}{x}=A$，对 $\frac{\varepsilon}{2}>0$，$\exists X_2>0$，$\forall x>X_2$，$\left|\frac{f(x)}{x}-A\right|<\frac{\varepsilon}{2}$.

取 $X=\max\{X_1,X_2\}$，$\forall x>X$，$\exists\xi\in(X,x)$，$f(x)=\frac{f(x)-f(X)}{x-X}x=f'(\xi)x$.

$|f(x)|=|f'(\xi)x|<\frac{\varepsilon}{2}x<\varepsilon$.

18. $\lim_{x\to 0}f(x)=0$，$\lim_{x\to 0}\frac{f(x)}{x}=0$.

解：$\forall\varepsilon>0$，对 $\frac{\varepsilon}{2}>0$，$\exists\delta>0$，$\forall x_1,x_2\in(0,\delta)$，$x_1<x_2$，$\exists\xi\in(x_1,x_2)$，$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)$.

$|f(x_2)-f(x_1)|=|f'(\xi)(x_2-x_1)|<\frac{\varepsilon}{2}(x_2-x_1)$.

$\lim_{x\to 0}\frac{f(x)}{x}=0$，$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$，$\lim_{x\to 0}f(x)=0$.

$\forall\frac{\varepsilon}{2}>0$，$\exists X_1>0$，$\forall x<X_1$，$\left|\frac{f(x)}{x}\right|<\frac{\varepsilon}{2}$.

$\lim_{x\to 0}\frac{f(x)}{x}=0$，$\exists X_2>0$，$\forall x<X_2$，$\left|\frac{f(x)}{x}\right|<\frac{\varepsilon}{2}$.

$\exists\xi\in(X_1,x)$，$f(x)=\frac{f(x)-f(0)}{x-0}x$，$f(x)=f'(\xi)(x-0)=f'(\xi)x$.

$\frac{f(x)}{x}=\frac{f'(\xi)x}{x}=f'(\xi)<\frac{\varepsilon}{2}<\varepsilon$.

19. $|f'(x)|\leq M$，$f(a)=f(b)=0$，$|f(c)|=|f(\frac{a+b}{2})|=A$.

$f(x)$ 在 $[a,\frac{a+b}{2}]$，$[\frac{a+b}{2},b]$ 上满足拉格朗日中值定理.

$\exists\xi\in(a,c)$，$f(c)=f'(\xi)(c-a)$，$|f(c)|=|f'(\xi)(c-a)|\leq M\cdot\frac{b-a}{2}$.

$\exists\eta\in(c,b)$，$f(b)-f(c)=f'(\eta)(b-c)$，$|f(b)-f(c)|=|f'(\eta)(b-c)|\leq M\cdot\frac{b-a}{2}$.

$|f(a)|+|f(b)|=|f'(\xi)(c-a)|+|f'(\eta)(b-c)|\leq|f'(\xi)(c-a)|+|f'(\eta)(b-c)|$

$\leq|f'(\xi)(c-a)|+|f'(\eta)(b-c)|\leq M\cdot\frac{b-a}{2}+M\cdot\frac{b-a}{2}=M(b-a)$.

20. $|f'(x)|<M$，$f(x_1)=f(x_2)$，$|f(x_0)-f(x_1)|<\frac{\varepsilon}{2}$.

$\left|\frac{f(x_0)-f(x_1)}{x_0-x_1}\right|<\frac{|f(x_0)-f(x_1)|}{|x_0-x_1|}<\frac{M}{2}$.

$\forall x_1\in[a,b]$，$\forall x_0\in[a,b]$，对 $\forall\varepsilon>0$，$\exists\delta$，

$\exists\xi\in(x_1,x_0)$，使 $f'(\xi)=\frac{f(x_0)-f(x_1)}{x_0-x_1}$，$|f'(\xi)|=\left|\frac{f(x_0)-f(x_1)}{x_0-x_1}\right|<M$.

21. $\lim_{x\to 0}\frac{e^{\sin x}-e^{\tan x}}{x^3}$

$=\lim_{x\to 0}\frac{e^{\tan x}(e^{\sin x-\tan x}-1)}{x^3}=\lim_{x\to 0}\frac{e^{\tan x}(e^{\sin x-\tan x}-1)}{x^3}$

$=\lim_{x\to 0}\frac{e^{\tan x}}{x^3}\cdot\frac{e^{\sin x-\tan x}-1}{\sin x-\tan x}\cdot\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}e^{\tan x}\cdot\frac{\sin x-\tan x}{x^3}$

$=\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}\frac{\sin x(\cos x-1)}{x^3\cos x}=\lim_{x\to 0}\frac{x\cdot\left(-\frac{1}{2}x^2\right)}{x^3\cdot 1}=-\frac{1}{2}$

$\lim_{x\to 0}\frac{\ln(1+x^2)-\ln(1+x)}{\sin^2 x}=\lim_{x\to 0}\frac{\ln\left(\frac{1+x^2}{1+x}\right)}{x^2}=\lim_{x\to 0}\frac{\ln\left(1+\frac{x^2-x}{1+x}\right)}{x^2}$

$=\lim_{x\to 0}\frac{\frac{x^2-x}{1+x}}{x^2}=\lim_{x\to 0}\frac{x-1}{1+x}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{x-1}{x(1+x)}=-\infty$

$\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln\cos bx}=\lim_{x\to 0}\frac{\ln(1+\cos ax-1)}{\ln(1+\cos bx-1)}=\lim_{x\to 0}\frac{\cos ax-1}{\cos bx-1}$

$=\lim_{x\to 0}\frac{-\frac{1}{2}a^2x^2}{-\frac{1}{2}b^2x^2}=\frac{a^2}{b^2}$

$\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)