补充题

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习题 详.

1. $n^n \gg n! \gg a^n \gg n^a \gg (\ln n)^m$

① $\lim\limits_{n\to\infty}\frac{n!}{n^n} = \lim\limits_{n\to\infty}\frac{n(n-1)\cdots 1}{n\cdot n\cdots n} < \frac{n\cdot n\cdots 1}{n\cdot n\cdots n} = \frac{1}{n}$

② $\lim\limits_{n\to\infty}\frac{a^n}{n!} = \lim\limits_{n\to\infty}\frac{a}{1}\times\frac{a}{2}\times\cdots\times\frac{a}{m}\times\frac{a}{m+1}\times\cdots\times\frac{a}{n}$

$\leq \lim\limits_{n\to\infty}\frac{a^m}{m!}\times 1\times 1\times\cdots\times\frac{a}{n} = 0$ $(a<m)$

2. $\lim\limits_{x\to0^+}(\cos\sqrt{x})^{\frac{1}{x}} = \lim\limits_{x\to0^+} e^{\frac{1}{x}\ln(\cos\sqrt{x})} = \lim\limits_{x\to0^+} e^{\frac{\ln(\cos\sqrt{x})}{x}}$

$= \lim\limits_{x\to0^+} e^{\frac{\ln(1-\frac{1}{2}x^2)}{x}} = \lim\limits_{x\to0^+} e^{\frac{-\frac{1}{2}x^2}{x}} = 1.$

$= \lim\limits_{x\to0^+}\left[(1+\cos\sqrt{x}-1)^{\frac{1}{\cos\sqrt{x}-1}}\right]^{\frac{\cos\sqrt{x}-1}{x}} = e^{\lim\limits_{x\to0^+}\frac{\cos\sqrt{x}-1}{x}} = \frac{1}{\sqrt{e}}.$

$\lim\limits_{x\to0}\frac{a^x-1}{x}$ $u=a^x$, $x=\log_a u$. $x\to0$, $u\to1$

$= \lim\limits_{u\to1}\frac{u-1}{\log_a u}$ 令 $u=x\ln a$. $a^x\to e^{x\ln a}=e^u$. $\frac{e^u-1}{u}\ln a = \ln a$.

$\lim\limits_{x\to0}\frac{(1+x)^a-1}{x} = \lim\limits_{x\to0}\frac{e^{a\ln(x+1)}-1}{x} = \lim\limits_{x\to0}\frac{a\ln(x+1)}{x} = a.$

10.11 习题.

1. 常数 $0<q<1$，并令 $a_n=\prod\limits_{k=1}^{n}(1+q^k)$，试证数列 $\{a_n\}_{n=1}^{\infty}$ 收敛.

证明：a_n 单调递增. → 只需证其有(上)界.

$a_n < \left(\frac{1+q+1+q^2+\cdots+1+q^n}{n}\right)^n$

$= \left(1+\frac{q}{n}\times\frac{1-q^n}{1-q}\right)^n < \left(1+\frac{q}{n(1-q)}\right)^n = b_n.$

$b_n = \left[\left(1+\frac{1}{n}\times\frac{q}{(1-q)}\right)^{n\times\frac{1-q}{q}}\right]^{\frac{q}{1-q}} \to e^{\frac{q}{1-q}}.$

即 $\lim\limits_{n\to\infty} b_n = e^{\frac{q}{1-q}} \triangleq b.$

思考：$b_n=\prod\limits_{k=1}^{n}(1-q^k)$，证明 $\lim\limits_{n\to\infty}b_n=b$，$b>0$.

2. 常数 $\alpha>0$，令 $x_n=\sum\limits_{k=1}^{n}\sin\frac{k\alpha}{n^2}$，试证 $\lim\limits_{n\to\infty}x_n=\frac{\alpha}{2}$.

证明：令 $y_n=\sum\limits_{k=1}^{n}\frac{k\alpha}{n^2}=\frac{(n+1)\alpha}{2n}$，$\lim\limits_{n\to\infty}y_n=\frac{\alpha}{2}$.

可拆为：$\lim\limits_{n\to\infty}x_n = \underbrace{\lim\limits_{n\to\infty}(x_n-y_n)}_{\text{证}\ 0} + \underbrace{\lim\limits_{n\to\infty}y_n}_{\frac{\alpha}{2}}$.

当整数 $n>\alpha$ 时，$0<\underbrace{\frac{k\alpha}{n^2}<\frac{\alpha}{n}}_{1\leq k\leq n}<1$ ⇒ $x>\sin x$. 估算.

$y_n-x_n=\sum\limits_{k=1}^{n}\left(\frac{k\alpha}{n^2}-\sin\frac{k\alpha}{n^2}\right) \leq n\left(\frac{n\alpha}{n^2}-\sin\frac{n\alpha}{n^2}\right)$

$=\alpha-n\sin\frac{\alpha}{n}$. $\therefore \lim\limits_{n\to\infty}|y_n-x_n|=0$

$0<y_n-x_n<\alpha-n\sin\frac{\alpha}{n}\to0.$ 夹逼

和比&夹比：$n\times\underbrace{\sin\frac{k\alpha}{n^2}}_{\min} < x_n < n\cdot\underbrace{\sin\frac{k\alpha}{n^2}}_{\max}$.

3. 常数 $0<\alpha<1$. 试证

(1) $\lim\limits_{x\to+\infty}[(x+c)^\alpha-x^\alpha]=0.$

(2) $\lim\limits_{x\to+\infty}[(x+\ln x)^\alpha-x^\alpha]=0$

(3) $0<\beta<1-\alpha$，$\lim\limits_{x\to+\infty}[(x+x^\beta)^\alpha-x^\alpha]=0$

增强.

$f(x)=(x+x^\beta)^\alpha-x^\alpha$ $\quad$ $t\to0$，$(1+t)^\alpha-1\sim\alpha t$.

$=x^\alpha\left[\left(1+\frac{1}{x^{1-\beta}}\right)^\alpha-1\right]\sim x^\alpha\cdot\alpha\cdot\frac{1}{x^{1-\beta}}=\frac{\alpha}{x^{1-\alpha-\beta}}.$

$1-\alpha-\beta>0$，$f(x)\to0.$

4. 设 $\lim\limits_{n\to\infty}a_n=a$，$b_n>0$，令 $A_n=\sum\limits_{k=1}^{n}a_kb_k$，$B_n=\sum\limits_{k=1}^{n}b_k$，试证 $\lim\limits_{n\to\infty}\frac{A_n}{B_n}$ 收敛.

证明：i) B_n 严格单增至 $+\infty$，

$\lim\limits_{n\to\infty}\frac{A_n}{B_n}=\lim\limits_{n\to\infty}\frac{A_n-A_{n-1}}{B_n-B_{n-1}}=\lim\limits_{n\to\infty}a_n=a.$

ii) $\lim\limits_{n\to\infty}B_n=B$. ($B>0$，$B$ 有限).

分母收敛，此时只需证分子收敛.

[可用 Cauchy 证明；A_n 有界].

$|A_{n+p}-A_n|=|a_{n+p}b_{n+p}+\cdots+a_{n+1}b_{n+1}|$

$\exists M$ 使 $|a_n|\leq M$. $\quad\leq M(b_{n+1}+\cdots+b_{n+p})$

对 $\forall\varepsilon>0$，$\exists N$，当 n，$n+p>N$ 时，有 $p\in N^*$

$|B_{n+p}-B_n|<\frac{\varepsilon}{M}$.

$\therefore |A_{n+p}-A_n|\leq M|B_{n+p}-B_n|=\varepsilon.$

5. ξ 无理数，$a,b\in\mathbb{R}$.

求证：存在整数 m,n 使 $m+n\xi\in(a,b)$.

即集合 $S=\{m+n\xi\,|\,m,n\in\mathbb{Z}\}$ 在 $\mathbb{R}$ 中稠密.

实数集 S 稠密：S 与任意非空开子集的交集非空. ((a,b) 可任意小，任意远).

递归定义一个严格单减的无理数列 x_n 如下：

$x_0=\underbrace{\xi-[\xi]}$，$\underbrace{x_{n+1}=1-\left[\frac{1}{x_n}\right]x_n}$，$n=0,1,2\cdots$

无-整=无. $\in(0,1)$. "尺子比偏"

进一步可验证 $\left\{\left[\frac{1}{x_n}\right]\right\}_{n=0}^{\infty}$ 严格单增至 $+\infty$. 证明.

$\therefore \lim\limits_{n\to\infty}x_n=0.$ $\left[\begin{array}{l}\frac{1}{x_n}>\left[\frac{1}{x_n}\right].\ \lim\limits_{n\to\infty}\left[\frac{1}{x_n}\right]=+\infty;\\ \text{若}\ x_n\to x\ (x>0)\text{，则无可得}\ \frac{1}{x_n}\to+\infty.\end{array}\right]$

可验证 (1) 对 $\forall x,y\in S$，$k\in\mathbb{Z}$，$x\pm y\in S$，$kx\in\mathbb{Z}$.

(2) $x_0=m+1\xi\in S$. $x_n=l+m\xi\in S$.

$x_{n+1}=1-k(l+m\xi)=1-kl-k\xi\in S$.

归纳法. $x_n\in S$. ，$n=0.1.2\ldots$

$\therefore$ 有 $x_n\in(0,1)\cap S$，$n=0,1\ldots$

x_n 无理，单减 $\to0$

对 $\forall(a,b)$，$\exists N$，$n>N$ 时，有 $0<x_n<b-a$.

取 x_{n_0}. $\quad\cdots\ -2x_{n_0}\ \ -x_{n_0}\ \ 0\ \ x_{n_0}\ \ 2x_{n_0}\ \cdots$

学习指导

1. $\tan x-\sin x\sim\frac{1}{2}x^3.$

$\lim\limits_{x\to0}\frac{\tan x-\sin x}{\frac{1}{2}x^3}=\frac{\tan x(1-\cos x)}{\frac{1}{2}x^3}=\frac{x\cdot\frac{1}{2}x^2}{\frac{1}{2}x^3}.$

2. (1) $\lim\limits_{x\to\infty}\left(\frac{4x-3}{4x+1}\right)^{3x}=\lim\limits_{x\to\infty}\left(1-\frac{4}{4x+1}\right)^{3x}=1$

$=\lim\limits_{x\to\infty}\underbrace{\left(1+\frac{-4}{4x+1}\right)^{\frac{4x+1}{-4}}}_{e}{}^{\underbrace{\frac{-4}{4x+1}\cdot3x}_{-3}}=e^{-3}.$

(2) $\lim\limits_{x\to0}(\cos x)^{\frac{1}{x\sin x}}=\lim\limits_{x\to0}e^{\frac{1}{x\sin x}\ln\cos x}$

$=e^{\lim\limits_{x\to0}\frac{1}{x\sin x}\ln\cos x}=e^{\lim\limits_{x\to0}\frac{\cos x-1}{x\sin x}}$

$=e^{\lim\limits_{x\to0}\frac{-\frac{1}{2}x^2}{x\sin x}}=\frac{1}{\sqrt{e}}$ $\quad\ln(x+1)\sim x$. $\ln x\sim x-1$.

(3) $\lim\limits_{x\to0}\left(\frac{\sin x}{\tan x}\right)^{\frac{1}{\tan^2 x}}=\lim\limits_{x\to0}e^{\frac{1}{\tan^2x}\ln\frac{\sin x}{\tan x}}=e^{\lim\frac{1}{\tan^2 x}\ln\frac{\sin x}{\tan x}}$

$=e^{\lim\limits_{x\to0}\frac{\ln\frac{\sin x}{\tan x}}{\tan^2x}}=e^{\lim\limits_{x\to0}\frac{\frac{\sin x}{\tan x}-1}{\tan^2 x}}=e^{\lim\frac{\sin x-\tan x}{\tan^3 x}}$

$=e^{\lim\frac{\cos x-1}{\tan^2 x}}=e^{\lim\frac{-\frac{1}{2}x^2}{\tan^2 x}}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}.$

3. (1) $\lim\limits_{x\to0}\frac{\ln(\sin x+e^{3x})-3x}{\ln(\tan x+e^{5x})-5x}=\lim\limits_{x\to0}\frac{\sin x+e^{3x}-1-3x}{\tan x+e^{5x}-1-5x}$ $\quad$ $3x\to\ln(e^{3x})$，$5x\to\ln(e^{5x})$

$=\lim\limits_{x\to0}\frac{\sin x+3x-3x}{\tan x+5x-5x}=\lim\limits_{x\to0}\frac{\sin x}{\tan x}=1$

(2) $\lim\limits_{x\to0}\frac{(1+x^2)^\pi-\cos x}{\sin x^2}=\lim\limits_{x\to0}\frac{e^{\pi\ln(x^2+1)}-\cos x}{\sin x^2}$

$=\lim\limits_{x\to0}\frac{e^{\pi x^2}-\cos x}{\sin x^2}=\lim\limits_{x\to0}\frac{\pi x^2+1-\cos x}{\sin x^2}=\lim\limits_{x\to0}\frac{\pi x^2+\frac{1}{2}x^2}{x^2}$

$=\lim\limits_{x\to0}\frac{\pi+\frac{1}{2}}{1}=\pi+\frac{1}{2}.$

(3) $\lim\limits_{x\to0}\frac{(1+ax)^b-(1+cx)^d}{x}=\lim\limits_{x\to0}\frac{e^{b\ln(1+ax)}-e^{d\ln(1+cx)}}{x}$

$=\lim\limits_{x\to0}\frac{e^{abx}-e^{cdx}}{x}=\frac{abx+1-cdx-1}{x}=ab-cd.$

(4) $\lim\limits_{x\to0}\frac{x}{\ln x}(x^x-1)=\lim\limits_{x\to0}\frac{x}{\ln x}(e^{x\ln x}-1)$

$=\lim\limits_{x\to0}\frac{x}{\ln x}(e^{\frac{\ln x}{x}}-1)=\lim\limits_{x\to0}\frac{x}{\ln x}\cdot\frac{\ln x}{x}=1\ ?$

4. (1) $[1+a_1t+o(t)][1+a_2t+o(t)]$

$=1+(a_1+a_2)t+\underbrace{2o(t)+o(t)(a_1+a_2)t+o(t)^2}_{\to o(t)}$

(2) $\lim\limits_{x\to0}\frac{\cos2x-e^{x^2}\cos x}{1-\cos x\cos2x\cos3x}$. $\quad e^t=1+t+o(t)$. $\cos t=1-\frac{1}{2}t^2+o(t^2)$.

$=\lim\limits_{x\to0}\frac{(1-\frac{1}{2}4x^2)-(1+x^2+o(x^2))(1-\frac{1}{2}x^2)}{1-(1-\frac{1}{2}x^2)(1-\frac{1}{2}4x^2)(1-\frac{1}{2}9x^2)}=\frac{1-2x^2+o(x^2)-(1+\frac{1}{2}x^2+o(x^2))}{1-(1-7x^2)}$

$=\lim\limits_{x\to0}\frac{-\frac{5}{2}x^2}{7x^2}=-\frac{5}{14}$

(3) $1-\prod\limits_{k=1}^{n}\sqrt[k]{\cos kx}=ax^2+o(x^2)$

$=1-\prod\limits_{k=1}^{n}\left(1-\frac{1}{2}k^2x^2\right)^{\frac{1}{k}}=1-\left(1-\frac{1}{2}\sum\limits_{k=1}^{n}k^2x^2+o(x^2)\right)$

$\sqrt[k]{\cos kx}=\sqrt[k]{1+(\cos kx-1)}=1+\frac{1}{k}(\cos kx-1)+o((\cos kx-1))$

$=1-\frac{1}{k}(1-\cos kx)+o(x^2)$ (消去"1")

$=1-\frac{1}{k}\cdot\frac{1}{2}k^2x^2+o(x^2)=1-\frac{k}{2}x^2+o(x^2)$

5. (1) $\arctan(\sqrt[3]{x^3+x}-x)\sim\frac{1}{3x}$. $x\to\infty$

$\lim\limits_{x\to\infty}\frac{\arctan(\sqrt[3]{x^3+x}-x)}{\frac{1}{3x}}=$

$(\sqrt[3]{x^3+x}-x)=x\left(\sqrt[3]{1+\frac{1}{x^2}}-1\right)\sim x\cdot\underbrace{\frac{1}{3}\cdot\frac{1}{x^2}}=\frac{1}{3x}.$

☆ $\sqrt[n]{1+x}=1+\frac{1}{n}t+o(t)$

$(\sqrt[3]{x^3+x}-x)=x\left(\sqrt[3]{1+\frac{1}{x^2}}-1\right)=x\left(e^{\frac{1}{3}\ln(1+\frac{1}{x^2})}-1\right)$

$=x\cdot\frac{1}{3}\ln\left(1+\frac{1}{x^2}\right)=x\cdot\frac{1}{3}\times\frac{1}{x^2}$

(2) $(x^\alpha+x^\beta)^\gamma-x^{\alpha\gamma}=x^{\alpha\gamma}\left((1+\underbrace{x^{\beta-\alpha}}_{\to0})^\gamma-1\right)$

$=x^{\alpha\gamma}\left(e^{\gamma\ln(1+x^{\beta-\alpha})}-1\right)=x^{\alpha\gamma}\cdot x^{\beta-\alpha}\cdot\gamma.$

(3) $\lim\limits_{x\to0}\frac{1}{x^2}\left(\underbrace{\sqrt{4+\frac{1}{x}f(x)}-2}_{\to0}\right)=\frac{3}{4}$，$f(x)\sim3x^2$

$\lim\limits_{x\to0}\frac{\sqrt{4+\frac{f(x)}{x}}-2}{x^2}=\frac{3}{4}$

6. (1) $\lim\limits_{x\to+\infty}(\sqrt{ax^2+2bx+c}-dx-e)=0$

$\sqrt{ax^2+2bx+c}-dx-e=\frac{ax^2+2bx+c-d^2x^2-e^2-2dex}{\sqrt{ax^2+2bx+c}+dx+e}$

$=\frac{(a-d^2)x^2+(2b-2de)x+c-e^2}{\cdots}$.

$\therefore a=d^2$，$b=de$，$c=e^2$.

(2) $\lim\limits_{x\to0}\frac{\sqrt{4(1+x)^3-3}-\alpha-\beta x-\lambda x^2}{x^2}$

$=\lim\limits_{x\to0}\frac{4(1+x)^3-3-\alpha^2-\beta^2x^2-\lambda^2x^4-2\alpha\beta x-2\alpha\lambda x^2-2\beta\lambda x^3}{x^2(\sqrt{4(1+x)^3-3}+\alpha+\beta x+\lambda x^2)}$

$=\lim\limits_{x\to0}\frac{(4-2\beta\lambda)x^3-\lambda^2x^4+(12-\beta^2-2\alpha\lambda)x^2+(12-2\alpha\beta)x+4-\alpha^2}{x^2(\quad)}$

$\alpha\beta=6$，$\alpha=\pm2$ ⇒ $\beta=\pm3$.

$12=\beta^2+2\alpha\lambda=9+4\lambda$. $\lambda=\frac{3}{4}$

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1. $f(x)=\lim\limits_{n\to\infty}\frac{x^{n+2}}{\sqrt{2^{2n}+x^{2n}}}$

$x=2$，$f(x)=\frac{2^{n+2}}{\sqrt{2\times2^{2n}}}=\frac{2^{n+2}}{\sqrt{2}\times2^n}=2\sqrt{2}$.

$0<x<2$，$f(x)=\frac{0}{\sqrt{+\infty+0}}=0$

$x>2$，$f(x)=\sqrt{\frac{x^{2n+4}}{2^{2n}+x^{2n}}}=\sqrt{\frac{x^4}{(\frac{2}{x})^{2n}+1}}=x^2.$

$\therefore f(x)=\begin{cases}0, & x\in[0,2)\\ 2\sqrt{2}, & x=2\\ x^2, & x>2\end{cases}$ 跳跃 / 可导

2. $0<\ln(1+t)<t$.

$\ln(1+t)=t-\frac{1}{2}t^2+o(t^2)$.

$a_{n+1}=\ln(a_n+1)$. $\lim\limits_{n\to\infty}na_n$.

$\lim\limits_{n\to\infty}na_n=\lim\limits_{n\to\infty}\frac{a_n}{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{\frac{1}{n+1}-\frac{1}{n}}$

$=\lim\limits_{n\to\infty}\frac{\ln(a_n+1)-a_n}{\frac{1}{n+1}-\frac{1}{n}}<\lim\limits_{n\to\infty}\frac{a_n-a_n}{\frac{1}{n+1}-\frac{1}{n}}=0$

$\lim\limits_{n\to\infty}\frac{a_n-\frac{1}{2}a_n^2-a_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{-\frac{1}{2}a_n^2}{\frac{1}{n+1}-\frac{1}{n}}=0$

倒用 $\lim\limits_{n\to\infty}\frac{n}{\frac{1}{a_n}}=\lim\limits_{n\to\infty}\frac{n+1-n}{\frac{1}{a_{n+1}}-\frac{1}{a_n}}$ → 消 n $=\lim\limits_{n\to\infty}\frac{a_na_{n+1}}{a_n-a_{n+1}}$

$=\lim\limits_{n\to\infty}\frac{a_n\ln(a_n+1)}{a_n-\ln(a_n+1)}=\lim\limits_{n\to\infty}\frac{a_n\cdot a_n}{a_n-(a_n-\frac{1}{2}a_n^2)}=2.$

$0<\ln(1+a_n)=a_{n+1}<a_n$. 单减有下界. → 收敛

$\lim\limits_{n\to\infty}a_n=0$.

3. $x_n=\arctan(\arctan(\arctan\cdots(\arctan(n!))))$

$0<\arctan(n!)<\frac{\pi}{2}$，$\lim\limits_{n\to\infty}\arctan(n!)=\frac{\pi}{2}$

x_n 单减，且 $x_n>0$ ⇒ 收敛.

$x_{n+1}=\arctan x_n$. $y=\arctan y$. $y=0$

令 $y_n=\arctan(\arctan\cdots(\arctan\frac{\pi}{2}))$，$y_1=\frac{\pi}{2}$.

$\therefore 0<x_n<y_n$，$n=1,2\ldots$

4. (1) $f(x)\in C[a,a+nh]$，$n\geq2$.

$\exists x_0\in[a,a+(n-1)h]$ 使 $f(x_0+h)-f(x_0)=\frac{f(a+nh)-f(a)}{n}$.

构造辅助 令 $F(x)=f(x+h)-f(x)$，$F(x)\in C[a,a+(n-1)h]$.

$F(a+(n-1)h)+F(a+(n-2)h)+\cdots+F(a)=f(a+nh)-f(a)$

$\frac{1}{n}\sum\limits_{k=1}^{n}F(a+(k-1)h)=\frac{f(a+nh)-f(a)}{n}$. "平均值"

$\min\{F(a+(k-1)h)\}\sim\max\{F(a+(k-1)h)\}$. 介于 max, min

(2) $g(x)$，$t=T$. $\exists x_0\in(-\infty,+\infty)$，$g(x_0+\frac{T}{n})=g(x_0)$

$g(x_0+\frac{1}{n})-g(x_0)$

$g(x_0+\frac{2}{n})-g(x_0+\frac{1}{n})$ $\quad$ $\lim\limits_{n\to+\infty}g(x_0+\frac{1}{n})-g(x_0)=0$

$\cdots$

$g(x_0+\frac{T}{n})-g(x_0+\frac{T-1}{n})$

5. $f(x)$ 在 $[a,+\infty)$ 连续，$f(+\infty)$ 存在且有限，$f(x)$ 在 $[a,+\infty)$ 一致连续.

$[\forall\varepsilon>0$，$\exists\delta$，$\forall x_0\in[a,+\infty)$，对 $\forall|x-x_0|<\delta$，$|f(x)-f(x_0)|<\varepsilon]$