课后题

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习题 13.1

$\ln x < x^m < x^n < e^x,\ x\to+\infty$

1. (1) $\frac{\ln(x^2+1)}{x}\geq\frac1x$，对 $\forall x>\sqrt{e-1}$.
$\int_0^{+\infty}\frac1x dx$ 发散，
$\therefore\int_0^{+\infty}\frac{\ln(x^2+1)}{x}dx$ 发散.

(3) $\lim_{x\to0}\frac{x\arctan x}{\sqrt[3]{1-x^4}}\to0$
$\lim_{x\to+\infty}\frac{x\arctan x}{\sqrt[3]{1+x^4}}=\frac\pi2\lim_{x\to+\infty}\frac{x}{x^{\frac43}}$
$\sim x^{-\frac13}=\frac{1}{x^{\frac13}}$，$\int_1^{+\infty}\frac{1}{x^{\frac13}}dx$ 发.

(5) $\lim_{x\to0}\frac{\ln x}{\sqrt{1-x^2}}$ 发散

(7) $\lim_{x\to0}\frac{e^{\sqrt x}-1}{\sqrt x}=\lim_{t\to0}\frac{e^t-1}{t}=1$
$\int_0^1\frac{1}{\sqrt x}dx=2\sqrt x$ 收敛.
$\therefore$ 收敛.

(9) $\lim_{x\to0}\frac{e^x-\cos x}{(1+x+\frac{x^2}{2}+\frac{x^3}{6})-(1-\frac{x^2}{2}+\frac{x^4}{24})}=1$
$\int_0^1\frac{1}{x+x^2}dx=\int_0^1\frac{1}{x(x+1)}dx$
$=\frac1x-\frac{1}{x+1}=\ln\frac{x}{x+1}\Big|_0^1$ 发.

(11) $\int_0^{\frac\pi2}\frac{d\sin x}{\sqrt{\sin x}}=\int_0^1\frac{dt}{\sqrt t}$ $\quad\frac{1}{\sqrt{\sin x}}\sim\frac{1}{\sqrt x}$
$=\frac12\sqrt t\Big|_0^1$ 收敛. 发散. $x\to0$.
$\frac{1}{\cos x}=\frac{1}{\sin(\frac\pi2-x)}\sim\frac{1}{\frac\pi2-x}$.

(13) $\lim_{x\to0}\frac{\frac{\ln\sin x}{\sqrt x}}{\frac{\ln x}{\sqrt x}}=\lim_{x\to0}\frac{\ln\sin x}{\ln x}=\frac{\frac{\cos x}{\sin x}}{\frac1x}$ $\quad x\to\frac\pi2$.
$x^\delta\ln|x|$ 有界，$\delta>0$，$x\to0$
$=\lim_{x\to0}\frac{x}{\sin x}=1$. （$\frac{\ln|y|}{y^\delta}$ 有界，$\delta>0$，$y\to+\infty$）
$\int_0^{\frac\pi2}\frac{\ln x}{\sqrt x}dx=2\int_0^{\frac\pi2}\ln x\,d\sqrt x$
$=4\int_0^{\sqrt{\frac\pi2}}\ln t\,dt=4(t\ln t-\int t\,d\ln t)$
$=4(\sqrt{\frac\pi2}\ln\sqrt{\frac\pi2}-\sqrt{\frac\pi2})$. 收敛.

(2) （$\sqrt{\frac{x}{e^x}}\le1$），$\frac{x}{e^x}\le1$，$x\le e^x$，恒成立）
对 $\forall x>0$，$\sqrt{\frac{x}{e^x}}\le1$ 恒成立，
$\therefore x^{\frac12}e^{-\frac12x}\le1$，$x^{\frac12}e^{-x}\le e^{-\frac12x}$.
$\int_0^{+\infty}e^{-\frac12x}dx$ 收敛，
$\therefore\int_0^{+\infty}\sqrt x e^{-x}dx$ 收敛.

(4) $\ln x=t$，$x\in[e^2,+\infty)$，$t\in[2,+\infty)$
$\int_2^{+\infty}\frac{1}{e^t\ln t}de^t=\int_2^{+\infty}\frac{1}{\ln t}dt$.
$\lim_{t\to+\infty}\frac{\frac{1}{\ln t}}{\frac1t}=\lim_{t\to+\infty}\frac{t}{\ln t}=\lim_{t\to+\infty}\frac{1}{\frac1t}=t\to+\infty$
$\int_2^{+\infty}\frac1t dt$ 发散，$\int_2^{+\infty}\frac{1}{\ln t}dt$ 发散.

(6) $t=1-x^2\in(0,1)$，$m=\frac1t\in(1,+\infty)$ $\quad$ 只看 $x\to1$
$\int_0^1\frac{1-t}{t^{\frac13}}d\sqrt{1-t}=\int_0^1\frac{1-t}{t^{\frac13}}\times\frac{-1}{2\sqrt{1-t}}dt$ $\quad\sim\frac{1}{\sqrt[3]{(1+x)^5(1-x)^5}}$
$=\int_0^1-\frac{\sqrt{1-t}}{t^{\frac13}}dt$ $\quad dt=d\frac1m=-\frac{1}{m^2}$ $\quad\sim\frac{1}{(1-x)^{\frac53}}$
$\lim_{t\to0}\frac{\frac{\sqrt{1-t}}{t^{1/3}}}{t^{-\frac13}}=1$，$\int_0^1t^{-\frac13}dt$ 与 $\int_0^1t^{-1}dt$ 敛

(8) $\int_0^1\frac{\sqrt x}{e^{\sin x}-1}dx$.
$\lim_{x\to0}\frac{x^{\frac12}}{e^{\sin x}-1}=\lim_{x\to0}\frac{\frac12x^{-\frac12}}{e^{\sin x}\cos x}=\frac12\lim_{x\to0}\frac{\sqrt x}{e^{\sin x}}=0$
$\frac{\frac{\sqrt x}{e^{\sin x}-1}}{\frac{1}{\sqrt x}}=\frac{x}{e^{\sin x}-1}$ $\quad e^{\sin x}\to1.\sim\frac{\sqrt x}{\sin x}\sim\frac{\sqrt x}{x}\sim\sqrt x$
$\int_0^1\frac{1}{\sqrt x}dx$ 收敛. $\therefore$ 收敛.

(12) $0\sim1$. $\mu>0$，$\arctan x\sim x$.
$\int_0^1\frac{1}{x^{\mu-1}}dx\Rightarrow\mu-1<1$ 收
$1\sim+\infty$，有界. $\arctan x<\frac\pi2$.
$\int_1^{+\infty}\frac{1}{x^\mu}dx\Rightarrow\mu>1$ 收
$\therefore\mu\in(1,2)$. 收敛

4. $f(y)-f(x)\le\int_x^yg(t)dt$. $\quad S_1$ $f(y)-\int_0^yg(t)dt\le f(x)-\int_0^xg(t)dt$ ⇓ $F(y)\le F(x)$ 单减.
设 $F(x)=\int_0^xg(t)dt$，则 $\lim_{x\to+\infty}F(x)$ 收敛，$F'(x)=g(x)$.
$g(x)$ 在 $[0,+\infty)$ 非负，$\therefore F(0)=0$，$F(x)\ge F(0)$.
$\therefore\exists M>0$，使对 $\forall x>0$，有 $F(x)\le M$. $\quad\therefore F(x)\ge-\int_0^{+\infty}g(t)dt>-M$.
$\therefore f(y)-f(x)\le F(y)-F(x)\le F(y)\le M$. $\quad\therefore\lim_{x\to+\infty}F(x)$ 存在.
$f(x)\ge0$，$\therefore$ 对 $\forall0<x<y$，$f(y)-f(x)\le f(y)\le M$. $\quad\therefore\lim_{x\to+\infty}f(x)$ 存在.
$\therefore f(x)$ 有界. $\quad S_2$ 假设不收敛
又对 $\forall x>0$，对 $\forall\varepsilon>0$，$\exists\delta$，当 $y-x<\delta m$，$\quad\exists\varepsilon>0$，对 $\forall M>0$，
$\int_x^yg(t)dt=F(\delta+x)-F(x)<\varepsilon$. $\quad\exists y>x>M$，（充分大）
即 $f(x+\delta)-f(x)<\varepsilon$. $\quad|f(y)-f(x)|>\varepsilon$.
$\therefore\lim_{x\to+\infty}f(x)$ 存在. $\quad\therefore f(y)-f(x)<-\varepsilon$.
取 $x_1<y_1<\cdots x_n<y_n$.
$f(y_n)-f(x_1)=f(y_n)-f(x_n)+f(x_n)-f(y_{n-1})\cdots$
$<-n\varepsilon+\sum_{i=1}^{n-1}f(x_{i+1})-f(y_i)$
$<-n\varepsilon+\sum_{i=1}^{n-1}\int_{y_i}^{x_{i+1}}g(t)dt$
$<-n\varepsilon+\int_0^{+\infty}g(t)dt$
与 $f(x)\ge0$ 矛盾.

5. $f(x)$ 非负，$\int_0^bf(x)dx=F(b)$ 单增.
$\int_0^{+\infty}f(x)g(x)dx$ 收敛. $\lim_{x\to+\infty}g(x)=0$.
对 $\forall\varepsilon>0$，$\exists X$，当 $x>X$ 时，$g(x)<\varepsilon$. $|\int_0^xf(t)g(t)d(t)-A|<\varepsilon$
$\int_a^bf(t)g(t)dt=g(a)\int_a^\xi f(t)dt+g(b)\int_\xi^bf(t)dt$. 收.
$g(x)(\int_0^Af(t)dt+\int_A^xf(t)dt)$ $\quad<\int_A^xg(t)f(t)dt<\frac\varepsilon2$
$=g(x)\int_0^Af(t)dt+g(x)\int_A^xf(t)dt$ $\quad$ 当取大 A，收敛（Cauchy）.
$f(x)$ 非负，$\exists M\ge0$ 使 $\int_0^Af(t)dt\le M$.
对 $\forall\varepsilon>0$，$\exists X$，当 $x>X$ 时，$g(x)<\frac{\varepsilon}{2M}$
$\therefore\lim_{x\to+\infty}g(x)\int_0^Af(t)dt<\frac\varepsilon2$.
$\int_0^xf(t)g(t)dt\le\int_0^xf(t)g(x)dt=g(x)\int_0^xf(t)dt$
$\lim_{x\to+\infty}\int_0^xf(t)g(t)dt=S$，$\lim_{x\to+\infty}\int_0^xf(t)dt\le\frac{S}{g(x)}$，$\int_0^xf(t)dt<\frac{S}{g(x)}$.
$\therefore$ 对 $\forall\varepsilon>0$，$\exists X$，当 $x>X$ 时，$g(x)<\frac{\varepsilon}{2\frac{S}{g(x)}}$.
$\therefore\lim_{x\to+\infty}g(x)\int_0^xf(t)dt<\frac\varepsilon2$.

6. 对 $\forall\varepsilon>0$，$\exists X$，当 $x>X$ 时，$|g(x)|<\varepsilon$.
$\int_0^xf(t)g(x-t)dt=\int_0^\xi f(t)g(x-t)dt+\int_\xi^xf(t)g(x-t)dt$
$\le\int_0^\xi|f(t)||g(x-t)|dt$ $\quad<\int_\xi^x|f(t)||g(x-t)|dt$
$<\frac{\varepsilon}{2M}\int_0^\xi|f(t)|dt$. $\quad<\frac{\varepsilon}{2M}\int_\xi^x|f(t)|dt$ （$g(x)$有界）$<L\int_\xi^x|f(t)|dt<L\frac{\varepsilon}{2L}$.
绝对收敛，$\int_0^\xi|f(t)|dt<M$.

2. (1) $\int_1^{+\infty}\cos(1-2x)dx$ 有界
$\frac{1}{\sqrt[3]x\sqrt[3]{x^2+1}}\sim\frac1x$，$\lim_{x\to\infty}\frac1x=0$.
$\therefore$ 本身收敛.
$\frac{|\cos(1-2x)|}{\sqrt[3]x\sqrt[3]{x^2+1}}\ge\frac{\cos^2(1-2x)}{\sqrt[3]x\sqrt[3]{x^2+1}}$
$=\frac12(1+\cos(2-4x))$.
$\frac{1}{2\sqrt[3]x\sqrt[3]{x^2+1}}$ 发散. 条件.

(3) 本身收敛.
$\int_2^{+\infty}\frac{1}{x\ln x}dx=\int_2^{+\infty}\frac{1}{\ln x}d\ln x$
$=\ln(\ln x)\Big|_2^{+\infty}$ 发散.
$\therefore$ 条件收敛.

(2) 同(1) 条件收敛. $\quad\frac{|\sin x|}{\sqrt[3]{x^2+x+1}}\ge\frac12\frac{1}{\sqrt[3]{x^2+x+1}}$
(4) $\frac{\ln(1+x)}{x^2}<\frac1x$，收敛.
$\left|\frac{\ln(1+x)}{x^2(1+x^p)}\right|<\left|\frac1x\right|$ 收敛.
$\therefore$ 绝对收敛. 发散，$\quad$ $x\to0$时. $\frac{\ln(1+x)}{x^2(1+x^p)}\sim\frac{x}{x^2}=\frac1x$
$I_n=\{x|\sin x>\frac12\}$ 不一定，无界

(6) $\lim_{x\to0}\frac{\frac{\sin x}{x^p}}{\frac{1}{x^{p-1}}}=1$.
$\int_0^1\frac{1}{x^{p-1}}dx$，$p-1<1$ 时收敛.
$\therefore p<0$时，绝对收敛.
$\int_0^1\frac{\sin\frac1x}{x^p}dx=\int_1^{+\infty}\frac{\sin y}{y^{2-p}}dy$.
$p<1$ 绝对
$p\in[1,2)$ 条件
$p\ge2$. 发散.

(5) 本身收敛.
$\int_1^{+\infty}\frac{1}{x(1+\sqrt x)}dx=\int_0^{+\infty}\frac1x-\frac{1}{(1+\sqrt x)\sqrt x}$
$=\int_0^{+\infty}\frac{1+\sqrt x-\sqrt x}{(1+\sqrt x)x}=\int_0^{+\infty}\frac1x-\frac{1}{\sqrt x}+\frac{1}{1+\sqrt x}dx$
发散. $\therefore$ 条件收敛.

习题 13.3

1. (1) $\lim_{\alpha\to0}\int_{-1}^1\sqrt{x^2+\alpha^2}dx=\int_{-1}^1\lim_{\alpha\to0}\sqrt{x^2+\alpha^2}dx=2\int_0^1x\,dx=1$.
$\int_{-1}^1\sqrt{x^2+\alpha^2}dx=\frac12(x\sqrt{x^2+\alpha^2}+\alpha^2\ln|x+\sqrt{x^2+\alpha^2}|)\Big|_{-1}^1$
$(\int\sqrt{x^2\pm a^2}dx=\frac12(x\sqrt{x^2\pm a^2}\pm a^2\ln|x+\sqrt{x^2\pm a^2}|)+C.)$
$=\frac12(\sqrt{1+\alpha^2}+\sqrt{1+\alpha^2}+\alpha^2\ln|\sqrt{1+\alpha^2}+1|-\alpha^2\ln|\sqrt{1+\alpha^2}-1|)$
$=\frac{\alpha^2}{2}\ln\frac{\sqrt{1+\alpha^2}+1}{\sqrt{1+\alpha^2}-1}\xrightarrow{\alpha\to0}\frac{\alpha^2}{2}(\ln2-\ln(\sqrt{1+\alpha^2}-1))$
$=\frac12\frac{\ln\frac{2}{\sqrt{1+\alpha^2}-1}}{\frac{1}{\alpha^2}}=\frac12\frac{\frac12(\sqrt{1+\alpha^2}-1)\times(-\frac{\alpha}{\sqrt{1+\alpha^2}})}{-2\frac{1}{\alpha^3}}=\frac18\frac{\alpha^4(\sqrt{1+\alpha^2}-1)}{\sqrt{1+\alpha^2}}\cdots$ $\quad$ 1.
$\alpha\to0$，$\sqrt{\alpha^2+1}+\alpha^2\ln(1+\sqrt{\alpha^2+1})-\frac{\alpha^2}{2}\ln\alpha^2=1$

(2) $\lim_{\alpha\to0}\int_\alpha^{1+\alpha}\frac{1}{1+x^2+\alpha^2}dx=\int_0^1\frac{1}{1+x^2}dx=\frac\pi4$.
$\int_\alpha^{1+\alpha}\frac{1}{x^2+(1+\alpha^2)}dx=\int_\alpha^{1+\alpha}\frac{1}{\frac{x^2}{\alpha^2+1}+1}dx\,\frac{1}{\alpha^2+1}$
$=\int_\alpha^{\alpha+1}\frac{1}{(\frac{x}{\sqrt{\alpha^2+1}})^2+1}d\frac{x}{\sqrt{\alpha^2+1}}\times\frac{1}{\sqrt{\alpha^2+1}}=\frac{1}{\sqrt{\alpha^2+1}}\int_{\frac{\alpha}{\sqrt{\alpha^2+1}}}^{\frac{\alpha+1}{\sqrt{\alpha^2+1}}}\frac{1}{t^2+1}dt$
$=\arctan t\Big|_{\frac{\alpha}{\sqrt{\alpha^2+1}}}^{\frac{\alpha+1}{\sqrt{\alpha^2+1}}}\times\frac{1}{\sqrt{\alpha^2+1}}$ $\quad\alpha\to0$，$1\times\arctan1=\frac\pi4$.

4. (1) $\int_0^{\frac\pi2}\ln(a^2\sin^2x+u^2\cos^2x)dx=F(u)$
$F'(u)=\int_0^{\frac\pi2}\frac{\cos^2x\,2u}{a^2\sin^2x+u^2\cos^2x}dx$
$=\int_0^{\frac\pi2}\frac{2u}{a^2\tan^2x+u^2}dx=\int_0^{\frac\pi2}\frac{2u}{\frac{a^2}{u^2}\tan^2x+1}\frac{1}{u^2}dx$
$\frac au\tan x=t$，$\frac uat=\tan x$，$\arctan\frac uat=x$ $\quad t\in[0,+\infty)$
$=\frac{1}{u^2}\int_0^{+\infty}\frac{1}{t^2+1}\frac{2u}{(\frac uat)^2+1}dt=\frac{2u}{u^2}\int_0^{+\infty}\frac{1}{(t^2+1)(a^2t^2+u^2)}dt$
$=\frac{2u}{u^2-a^2}\int_0^{+\infty}(\frac{1}{1+t^2}-\frac{a^2}{a^2t^2+u^2})dt=\frac{\pi}{u+a}$.
$F(b)-F(a)=\int_a^bF'(u)du=\pi\ln(a+b)-\pi\ln2a$
$F(a)=\int_0^{\frac\pi2}\ln a^2dx=\pi\ln a$. $\quad F(b)=\pi\ln(a+b)-\pi\ln2$.

(2) $\int_0^\pi\ln(1-2u\cos x+u^2)dx=F(u)$
$F'(u)=\int_0^\pi\frac{2u-2\cos x}{1-2u\cos x+u^2}dx$ $\quad\int_0^\pi\frac{1}{A-\cos x}dx$
$=\int_0^\pi\frac{u-2\cos x+\frac1u-\frac1u+u}{u^2-2u\cos x+1}dx$ $\quad\int_0^\pi\frac{1}{A-\frac{1-t^2}{1+t^2}}dx$
$=\int_0^\pi\frac1u+\frac{u-\frac1u}{u^2+1-2u\cos x}dx$ $\quad=\int_0^{+\infty}\frac{1}{A-\frac{1-t^2}{1+t^2}}\times\frac{2}{1+t^2}dt$
$=\frac\pi u+\frac{u-\frac1u}{2u}\int_0^\pi\frac{1}{\frac{u^2+1}{2u}-\cos x}dx$ $\quad=\int_0^{+\infty}\frac{1}{A-1+t^2(A+1)}dt$
$=\frac\pi u+\frac{u^2-1}{u(u-1)^2}\times\frac\pi2$ $\quad\frac{u^2-1}{u}\int_0^{+\infty}\frac{1}{1-2u\cos x+u^2}dx$ $\quad\arctan\Big|_0^{+\infty}$
$=\frac\pi u+\frac{u+1}{u(u-1)}\times\frac\pi2$ $\quad\frac\pi u+\frac{\pi}{1-u^2}\frac{u^2-1}{u}=0$
$=\frac{2\pi(u-1)+\pi(u+1)}{2u(u-1)}=\frac{\pi(3u-1)}{2u(u-1)}$
$F(a)=F(a)-F(0)=\int_0^a\frac\pi u(\frac{u+1}{2(u-1)}+1)du$
$=\int_0^a\frac\pi u+\frac{1}{u-1}\times\frac\pi2+\frac{1}{u(u-1)}\times\frac\pi2du$
$F(a)=F(0)=0$.
$a>1$，$|\frac1a|<1$，$F(a)=\int_0^\pi\ln(1-2a\cos x+a^2)dx$
$=\int_0^\pi\ln a^2(1-\frac2a\cos x+\frac{1}{a^2})dx$
$=2\pi\ln|a|+\int_0^\pi\ln(1-\frac2a\cos x+(\frac1a)^2)dx=2\pi\ln|a|$

(3) $\int_0^{\frac\pi2}\frac{\arctan(u\tan x)}{\tan x}dx=F(u)$
$F'(u)=\int_0^{\frac\pi2}\frac{1}{\tan x}\times\frac{1}{u^2\tan^2x+1}\times\tan x\,dx=\int_0^{\frac\pi2}\frac{1}{u^2\tan^2x+1}dx$
$=\int_0^{+\infty}\frac{1}{(u^2t^2+1)}\times\frac{1}{t^2+1}dt=\int_0^{+\infty}\frac{u^2}{u^2t^2+1}-\frac{1}{t^2+1}dt\,\frac{1}{(u^2-1)}$
$=\frac{1}{u^2-1}(u\arctan ut-\arctan t)\Big|_0^{+\infty}$
$=\frac{1}{u^2-1}(\frac\pi2u^2-\frac\pi2)=\frac\pi2\frac{u^2-1}{u^2+1}$
$F(a)-F(0)=F(a)=\int_0^aF'(u)du=\int_0^a\frac\pi2\frac{u^2-1}{u^2+1}du$
$=\frac\pi2\int_0^a1-\frac{2}{u^2+1}du=\frac\pi2(u-2\arctan u)\Big|_0^a$
$=\frac\pi2(a-2\arctan a)$ $\quad F(a)=\frac\pi2\ln(1+a)$

(4) $\int_0^{\frac\pi2}\ln\frac{1+u\cos x}{1-u\cos x}\frac{dx}{\cos x}=F(u)$ $\quad\frac{u\cos x-1+2}{1-u\cos x}=-1+\frac{2}{1-u\cos x}$
$F'(u)=\int_0^{\frac\pi2}\frac{1}{\cos x}\frac{1-u\cos x}{1+u\cos x}\times\frac{2\cos x}{(1-u\cos x)^2}dx$
$=-\int_0^{\frac\pi2}\frac{1}{1-u^2\cos^2x}dx=\int_0^{\frac\pi2}\frac{1}{u^2\cos^2x-1}dx$
$=\int_0^{\frac\pi2}\frac{\cos^2x+\sin^2x}{u^2\cos^2x-\cos^2x-\sin^2x}dx=\int_0^{\frac\pi2}\frac{1}{(u^2-1)\cos^2x-\sin^2x}dx$
$=\int_0^{\frac\pi2}\frac{\cos^2x}{(u^2-1)\cos^2x-\sin^2x}\times\frac{1}{\cos^2x}dx$ $\quad t^2+1-u$
$=\int_0^{\frac\pi2}\frac{1}{(u^2-1)-\tan^2x}d\tan x=\int_0^{+\infty}\frac{1}{u^2-1-t^2}dt$
$=-\int_0^{+\infty}\frac{1}{t^2+1-u^2}dt=-\int_0^{+\infty}\frac{1}{(\frac{t}{\sqrt{1-u^2}})^2+1}\times\frac{1}{1-u^2}dt$
$=-\int_0^{+\infty}\frac{1}{m^2+1}dm\times\frac{1}{\sqrt{1-u^2}}=-\frac{1}{\sqrt{1-u^2}}\arctan m\Big|_0^{+\infty}=-\frac\pi2\frac{1}{\sqrt{1-u^2}}$.
$F(a)=F(a)-F(0)=\int_0^a-\frac\pi2\frac{1}{\sqrt{1-u^2}}du$
$=-\frac\pi2\arcsin a$. $\quad\pi\arcsin a$.

2. (1) $F'(\alpha)=\int_{\sin\alpha}^{\cos\alpha}\sqrt{1-x^2}e^{\alpha\sqrt{1-x^2}}dx-e^{\alpha\sin\alpha}\sin\alpha-e^{\alpha\cos\alpha}\cos\alpha$
emm 然后咋算

(2) $F'(\alpha)=\int_{a+\alpha}^{b+\alpha}\frac1x\cos\alpha x\cdot x\,dx+\frac{\sin\alpha(b+\alpha)}{b+\alpha}-\frac{\sin\alpha(a+\alpha)}{a+\alpha}$
$=\frac1\alpha\sin\alpha x\Big|_{a+\alpha}^{b+\alpha}+\frac{\sin\alpha(b+\alpha)}{b+\alpha}-\frac{\sin\alpha(a+\alpha)}{a+\alpha}$.
$=(\frac1\alpha+\frac{1}{b+\alpha})\sin\alpha(b+\alpha)-(\frac1\alpha+\frac{1}{a+\alpha})\sin\alpha(a+\alpha)$.

(3) $F'(\alpha)=\int_0^\alpha\frac1x\frac{1}{(1+\alpha x)}x\,dx+\frac{\ln(1+\alpha^2)}{\alpha}$
$=\int_0^\alpha\frac{1}{\alpha x+1}dx+\frac{\ln(1+\alpha^2)}{\alpha}$ $\quad\frac1\alpha\ \frac{1}{x+\frac1\alpha}$
$=\frac1\alpha\ln(x+\frac1\alpha)\Big|_0^\alpha+\frac1\alpha\ln(1+\alpha^2)$
$=\frac1\alpha(\ln(\alpha+\frac1\alpha)+\ln(\alpha^2+1)-\ln\frac1\alpha)$
$=\frac1\alpha(\ln(\alpha^2+1)-\ln\alpha+\ln(\alpha^2+1)+\ln\alpha)$
$=\frac2\alpha\ln(\alpha^2+1)$

(4) $g(x,\alpha)=f(x+\alpha,x-\alpha)$. $\frac{\partial g}{\partial\alpha}=f'_u-f'_v$
$F(\alpha)=\int_0^\alpha f'_u-f'_v\,dx+f(2\alpha,0)$

3. $y(x)=\frac1k\int_c^xf(t)\sin k(x-t)dt$
$y'(x)=\frac1k\int_c^xf'(t)\sin k(x-t)-kf(t)\cos k(x-t)dt$
$=\frac1k\int_c^xf'(t)\sin k(x-t)dt-\int_c^xf(t)\cos k(x-t)dt$.
$=\frac1k\int_c^x\sin k(x-t)df(t)-\int_c^xf(t)\cos k(x-t)dt$
$=\frac1kf(t)\sin k(x-t)\Big|_c^x-\frac1k\int_c^xf(t)\cos k(x-t)dt\times(-k)$
$-\int_c^xf(t)\cos k(x-t)dt$
$=\frac1kf(x)\sin k(0)-\frac1kf(c)\sin k(x-c)$
$=-\frac1kf(c)\sin k(x-c)$
$y''(x)=-f(c)\cos k(x-c)$

$k^2y(x)=k\int_c^xf(t)\sin k(x-t)dt$
$=\int_c^xf(t)d\cos k(x-t)$
$=f(t)\cos k(x-t)\Big|_c^x-\int_c^xf'(t)\cos k(x-t)dt$
$=f(x)-f(c)\cos k(x-c)-\int_c^xf'(t)\cos k(x-t)dt$

$y(x)=\frac1k\int_c^xf(t)\sin k(x-t)dt$. $\quad g(x,t)=f(t)\sin k(x-t)$
$y'(x)=\frac1k\int_0^xf(t)(k)\cos k(x-t)dt+\frac1kf(x)\sin k(x-x)\times1$
$-\frac1kf(c)\sin(x-c)\times0$
$=\int_0^xf(t)\cos k(x-t)dt$.
$y''(x)=[\int_c^xf(t)(-k)\sin k(x-t)(1)dt+f(x)\cos k(x-x)$ ~~dt~~
$-f(c)\cos k(c-t)\times0]$
$=-[k\int_c^xf(t)\sin k(x-t)dt+f(x)]$
$=-k\int_c^xf(t)\sin k(x-t)dt+f(x)$.
$y''(x)+k^2y(x)=k\int_0^xf(t)\sin k(x-t)dt-\cdots$

习题 13.4

1. (1) $\int_0^{+\infty}x^udx=\frac{1}{u+1}x^{u+1}\Big|_0^{+\infty}$ $\quad(u\ne-1)$
则 $u<-1$ 时收敛.
$u=-1$ 不收敛.
收敛域 $(-\infty,-1)$

(3) $\int_2^{+\infty}\frac{1}{x^u\ln x}dx$
$=\int_2^{+\infty}\frac{1}{x^{u-1}\ln x}d\ln x$
$u=1$，$=\ln(\ln x)\Big|_2^{+\infty}$
$\int_2^{+\infty}\frac{1}{x^2\ln x}dx$
$=\int_2^{+\infty}\frac{?}{\ln x}d(\frac1x)$

(5) $x\to0$. $\frac{\sin x}{x^\alpha+x^{\alpha+1}}\sim\frac{\sin x}{x^\alpha}$
$\alpha=2$

(2) $\lim_{x\to+\infty}\frac{x+\sin x}{x-\sin x}=1$.
$x\to+\infty$，$x^u\frac{x+\sin x}{x-\sin x}\sim x^u$.
$\int_1^{+\infty}x^udx$ 收敛，$u\in(-\infty,-1)$.

(4) $t=\sin x$，$dx=\frac{1}{\sqrt{1-t^2}}dt$
$\int_0^1\frac{1}{t^u\sqrt{1-t^2}}dt$
$=\int_0^1\frac{1}{t^u}d\arcsin t$
$=\frac{\arcsin t}{t^u}\Big|_0^1+\int_0^1\arcsin t\,u\frac{1}{t^{u+1}}dt$
$\int_0^1\frac{1}{t^{u+1}}dt$ 收敛，$u<0$. $\quad\frac{\arcsin t}{t^u}$ 收敛. $u<0$

(6) $\int_0^{+\infty}\frac{\ln(1+x^2)}{x^\alpha}dx$. $\int_0^1\int_1^{+\infty}$
$x\to0$，$\int_0^{+\infty}\frac{\ln(1+x^2)}{x^2}\times\frac{x^2}{x^\alpha}dx$
$\frac{\ln(1+x^2)}{x^2}=1$. $\int_0^1\frac{1}{x^{\alpha-2}}dx$.
$\alpha-2<1$，$\alpha<3$.
$x\to+\infty$，$\int_1^{+\infty}\frac{\ln(1+x^2)}{x^\alpha}dx$ $\quad\Rightarrow(1,3)$.
$\frac{2x}{\alpha x^{\alpha-1}(1+x^2)}\sim\frac2\alpha\frac{1}{x^\alpha}$
收敛，$\alpha>1$.

2. (1) $\int_{-\infty}^{+\infty}\frac{\cos ux}{1+x^2}dx$. $u\in(-\infty,+\infty)$.
$\left|\frac{\cos ux}{1+x^2}\right|\le\frac{1}{1+x^2}$
$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx$ 收敛.
$\therefore$ 一致收敛.

(3) $\int_0^{+\infty}\frac{\sqrt\alpha}{e^{\alpha x^2}}dx$
$\frac{\sqrt\alpha}{e^{\alpha x^2}}\le\frac{\sqrt\alpha}{1+\alpha x^2}$

(5) $\int_1^{+\infty}e^{-\alpha x}\frac{\cos x}{x^p}dx$.
$g(x,\alpha)=\frac{1}{x^p}\downarrow\to0$.
$\int_1^Ae^{-\alpha x}\cos x\,dx$
$=\int_1^Ae^{-\alpha x}d\sin x$
$=e^{-\alpha x}\sin x\Big|_1^A+\alpha\int_1^A\sin x\,e^{-\alpha x}dx$
$=e^{-\alpha x}\sin x\Big|_1^A-\alpha\int_1^Ae^{-\alpha x}d\cos x$
$=e^{-\alpha x}\sin x\Big|_1^A-\alpha(e^{-\alpha x}\cos x\Big|_1^A$
$+\alpha\int_1^Ae^{-\alpha x}\cos x\,dx$
$=e^{-\alpha x}(\sin x\Big|_1^A-\alpha\cos x\Big|_1^A)$
$-\alpha^2\int_1^Ae^{-\alpha x}\cos x\,dx$
$\int_1^Ae^{-\alpha x}\cos x\,dx=\frac{\sin A-\sin1-\alpha\cos A+\alpha\cos1}{(1+\alpha^2)e^{\alpha x}}$
$\le\frac{\sin A+\alpha\cos1}{(1+\alpha^2)e^{\alpha x}}$
对 $\forall\alpha\ge0$ 有界. ?
$\therefore$ 一致收敛.

(2) $\int_0^{+\infty}e^{-\alpha x}\sin\beta x\,dx$
$g(x,\alpha)=\sin\beta x$ 一致有界.
$|\int_0^{+\infty}e^{-\alpha x}dx|=|-\frac1\alpha e^{-\alpha x}\Big|_0^{+\infty}|$
$=|-\frac1\alpha e^{-\alpha x}\Big|_0^{+\infty}|\le\frac1\alpha$.
$0<\alpha_0\le\alpha$，$\frac1\alpha\le\frac{1}{\alpha_0}$ 收.
$\therefore$ 此时一致收敛.
$0<\alpha<+\infty$，
$e^{-\alpha x}$ 连续，$\alpha=0$，$\int_0^{+\infty}1dx$ 发散.
$\therefore$ 不一致收敛.

(4) $\int_1^{+\infty}\frac{\ln(1+x^2)}{x^\alpha}dx$，$\quad\frac{\ln(1+x^2)}{x^2}$，$\frac{1}{x^{\alpha-2}}$
$\frac{\ln(1+x^2)}{x^\alpha}$ 对 $\forall\alpha\in(1,+\infty)$，$>0$ $\quad\int$ 收.
在 $x\in[1,+\infty)$ 连续.
$\alpha=1$，$\int_1^{+\infty}\frac{\ln(1+x^2)}{x}dx$
$=\int_1^{+\infty}\ln(1+x^2)d\ln x$
$=\ln(1+x^2)\ln x\Big|_1^{+\infty}-\int_1^{+\infty}\ln x\frac{2x}{x^2+1}dx$
发散，不一致收敛.

(6) $\int_0^{+\infty}\frac{\sin(x^2)}{1+x^p}dx$.
$g(x,p)=\frac{1}{1+x^p}$，对 $\forall p\ge0$，
在 $x\ge0$ 上一致有界.
$\int_0^{+\infty}\sin x^2dx=\int_0^{+\infty}\sin t\times\frac{1}{2\sqrt t}dt$
$=\int_0^{+\infty}\frac{\sin t}{2\sqrt t}dt$ 收敛.
$\therefore$ 一致收敛.

4. 对 $\forall\varepsilon>0$，$\exists\delta=$ ___ . $|u-u_0|<\varepsilon$ 时，
$\varphi(u)$ 连续，$\left|\int_a^A\frac{u}{e^{ux}}-\frac{u_0}{e^{u_0x}}dx\right|<\frac\varepsilon3$.
$\frac{u}{e^{ux}}$ 在 $x\in[A,+\infty)$ 可积，$\int_A^{+\infty}\frac{u}{e^{ux}}dx=-\frac{1}{e^{ux}}\Big|_A^{+\infty}\le\frac{1}{e^{uA}}$.
对 $\forall\varepsilon>0$，$\exists X>a$，当 $A>X$ 时，$\frac{1}{e^{uA}}<\frac\varepsilon3$，$\frac{1}{e^{u_0A}}<\frac\varepsilon3$.
$|\varphi(u)-\varphi(u_0)|=|\int_a^{+\infty}ue^{-ux}dx-\int_a^{+\infty}u_0e^{-u_0x}dx|$
$=|\int_a^Aue^{-ux}dx+\int_A^{+\infty}ue^{-ux}dx-\int_a^Au_0e^{-u_0x}dx-\int_A^{+\infty}u_0e^{-u_0x}dx|$
$\le|\int_a^Aue^{-ux}dx-\int_a^Au_0e^{-u_0x}dx|+|\int_A^{+\infty}ue^{-ux}dx|+|\int_A^{+\infty}u_0e^{-u_0x}dx|$
$\therefore$ 原式 $\le\frac\varepsilon3+\frac\varepsilon3+\frac\varepsilon3=\varepsilon$. 连续.

7. (1) $\int_0^1\frac{x^\beta-x^\alpha}{\ln x}dx$. $(\alpha,\beta>-1)$. $\quad\frac{\beta x^{\beta-1}-\alpha x^{\alpha-1}}{\frac1x}=\beta x^\beta-\alpha x^\alpha$ 存在. $\to1$ 收敛
$I(\beta)=\int_0^1\frac{x^\beta-x^\alpha}{\ln x}dx$. 连续. $\quad\beta>-1$ 一致收敛，
$I'(\beta)=\int_0^1\frac{x^\beta\ln x}{\ln x}dx=\int_0^1x^\beta dx=\frac{1}{\beta+1}x^{\beta+1}=\frac{1}{\beta+1}$.
$I(\beta)=\ln(\beta+1)+C$. $I(\alpha)=\ln(\alpha+1)+C=0$.
$\therefore I(\beta)=\ln\frac{\beta+1}{\alpha+1}$.

(2). $I(\alpha)=\int_0^{+\infty}\frac{1-e^{-\alpha x}}{xe^x}dx=\int_0^{+\infty}\frac{1}{xe^x}-\frac{e^{-\alpha x}}{xe^x}dx$ 收敛
$I'(\alpha)=\int_0^{+\infty}x\cdot\frac{e^{-\alpha x}}{xe^x}dx=\int_0^{+\infty}e^{-(\alpha+1)x}dx=-\frac{1}{(\alpha+1)}e^{-(\alpha+1)x}\Big|_0^{+\infty}$
$=\frac{1}{\alpha+1}e^{-(\alpha+1)x}\Big|_{+\infty}^0=\frac{1}{\alpha+1}$. $\quad$ 一致收敛.
$I(\alpha)=\ln(\alpha+1)+C$. $I(0)=0$. $I(\alpha)=\ln(\alpha+1)$.

(3). $I(\alpha)=\int_0^{+\infty}\frac{1-e^{-\alpha x^2}}{x^2}dx=\int_0^{+\infty}\frac{1}{x^2}-\frac{e^{-\alpha x^2}}{x^2}dx$
$I'(\alpha)=\int_0^{+\infty}-x^2\times(-\frac{e^{-\alpha x^2}}{x^2})dx=\int_0^{+\infty}e^{-\alpha x^2}dx$ $\quad\alpha\ge\alpha_0>0$ 一致.
$=\frac{1}{\sqrt\alpha}\int_0^{+\infty}e^{-(\sqrt\alpha x)^2}d(\sqrt\alpha x)=\frac12\sqrt{\frac\pi\alpha}$. $\quad(\sqrt x)'=\frac{1}{2\sqrt x}$.
$I(\alpha)=\sqrt{\pi\alpha}+C$. $I(0)=0$. $I(\alpha)=\sqrt{\pi\alpha}$.

(4). $I(\alpha)=\int_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}dx$ 连续. 收敛.
$I'(\alpha)=\int_0^{+\infty}\frac1x(-x^2)e^{-\alpha x^2}dx$ $\quad\alpha\ge\alpha_0>0$ 一致.
$=\int_0^{+\infty}-xe^{-\alpha x^2}dx=-\frac12\int_0^{+\infty}e^{-\alpha x^2}dx^2$.
$=-\frac{1}{2\alpha}\int_0^{+\infty}e^{-\alpha x^2}d(\alpha x^2)$
$=-\frac{1}{2\alpha}\int_0^{+\infty}e^{-t}dt$
$=-\frac{1}{2\alpha}(-e^{-t})\Big|_0^{+\infty}$；$\frac{1}{2\alpha}e^{-t}\Big|_0^{+\infty}=-\frac{1}{2\alpha}$.
$I(\alpha)=-\frac12\ln\alpha+C$. $I(\beta)=-\frac12\ln\beta+C=0$
$I(\alpha)=-\frac12\ln\frac\alpha\beta$.

(5) $I(a)=\int_0^{+\infty}\frac{\arctan ax}{x(1+x^2)}dx$.
$I'(a)=\int_0^{+\infty}x\cdot\frac{1}{x(1+x^2)}\frac{1}{(1+a^2x^2)}dx$ $\quad<\frac{1}{(1+x^2)}$ 一致.
$=\frac{1}{a^2-1}\int_0^{+\infty}\frac{1}{(1+x^2)}-\frac{1}{(1+a^2x^2)}dx$
$=\frac{1}{a^2-1}(\arctan x-\frac1a\arctan\frac xa)\Big|_0^{+\infty}$
$=\frac{1}{a^2-1}(\frac\pi2-\frac1a\frac\pi2)=\frac\pi2\frac{1}{a^2-1}\times\frac{a-1}{a}=\frac{\pi}{a(a+1)2}$
~~$=\frac\pi2(\frac1a-\frac{1}{a+1})$~~ $\quad\frac{\pi}{2(a+1)}$. $\quad I(a)=\frac\pi2\ln(a+1)$.
$I(a)=\frac\pi2\ln\frac{a}{a+1}+C$. $I(0)=0$ $\quad\cdots$?
$a\to+\infty$，$\int_0^{+\infty}\frac\pi2\frac{1}{x(1+x^2)}$

(6) $I(a)=\int_0^{+\infty}[e^{-(\frac ax)^2}-e^{-(\frac bx)^2}]dx$
$I'(a)=\int_0^{+\infty}-\frac{1}{x^2}e^{-(\frac ax)^2}\times2a\,dx$
$=-2\int_0^{+\infty}e^{-(\frac ax)^2}d\frac ax=-\sqrt\pi$.
$I(a)=-\sqrt\pi a+C$. $I(b)=0$.
$I(a)=-\sqrt\pi(a-b)$.

8. (1) $\int_{-\infty}^{+\infty}\frac{x}{\sigma\sqrt{2\pi}}e^{-\frac12(\frac{x-a}{\sigma})^2}dx$
$I(\sigma)=\int_{-\infty}^{+\infty}\frac{x}{\sqrt{2\pi}}\frac1\sigma e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}}dx$
$I'(\sigma)=\int_{-\infty}^{+\infty}\frac{x}{\sqrt{2\pi}}(-\frac{1}{\sigma^2}e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}}+\frac1\sigma(-\frac{(x-a)^2}{2})(-2\frac{1}{\sigma^3})e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}})dx$
$=\int_{-\infty}^{+\infty}\frac{x}{\sqrt{2\pi}}(-\frac{1}{\sigma^2}e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}}+\frac{1}{\sigma^4}(x-a)^2e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}})dx$
$=\int_{-\infty}^{+\infty}-\frac{x}{\sqrt{2\pi}}\frac{1}{\sigma^2}e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}}dx+\int_{-\infty}^{+\infty}\frac{x(x-a)^2}{\sqrt{2\pi}}\frac{1}{\sigma^4}e^{-\frac{(x-a)^2}{2}\frac{1}{\sigma^2}}dx$
$\int_{-\infty}^{+\infty}\frac{x}{\sigma\sqrt{2\pi}}e^{-\frac12(\frac{x-a}{\sigma})^2}dx$
$=\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}\times\frac12e^{-\frac12(\frac{x-a}{\sigma})^2}dx^2$
$=\frac{1}{2\sqrt{2\pi}\sigma}e^{-\frac12(\frac{x-a}{\sigma})^2}x^2\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\frac{x^2}{2\sigma\sqrt{2\pi}}de^{-\frac12(\frac{x-a}{\sigma})^2}$
$=-\int_{-\infty}^{+\infty}\frac{x^2}{2\sigma\sqrt{2\pi}}(-\frac12)2(\frac{x-a}{\sigma})\frac1\sigma e^{-\frac12(\frac{x-a}{\sigma})^2}dx$
$=\int_{-\infty}^{+\infty}\frac{x^2(x-a)}{2\sigma^3\sqrt{2\pi}}e^{-\frac12(\frac{x-a}{\sigma})^2}dx$

(2) $\int_{-\infty}^{+\infty}\frac{(x-a)^2}{\sigma\sqrt{2\pi}}e^{-\frac12(\frac{x-a}{\sigma})^2}dx$ $\quad e^{-(\frac{x-a}{\sqrt2\sigma})^2}$
$=\int_{-\infty}^{+\infty}\frac{(x-a)}{\sqrt2\sigma}\frac{(x-a)}{\sqrt\pi}e^{-(\frac{x-a}{\sqrt2\sigma})^2}dx$ $\quad\frac{x-a}{\sqrt2\sigma}=t$. $(x-a)=\sqrt2\sigma t$
$=\frac12\int_{-\infty}^{+\infty}\frac{x-a}{\sqrt\pi}e^{-(\frac{x-a}{\sqrt2\sigma})^2}d(\frac{x-a}{\sqrt2\sigma})^2$.
$=\frac12\int_{-\infty}^{+\infty}\frac{\sqrt2\sigma}{\sqrt\pi}te^{-t^2}dt^2$
$=\frac{\sqrt2\sigma}{2\sqrt\pi}t^3e^{-t^2}\Big|_{-\infty}^{+\infty}-\frac12\int_{-\infty}^{+\infty}t^2\frac{\sqrt2\sigma}{\sqrt\pi}(e^{-t^2}-2t^2e^{-t^2})dt$
$=\frac{\sqrt2\sigma}{2\sqrt\pi}\int_{-\infty}^{+\infty}2t^4e^{-t^2}-t^2e^{-t^2}dt$

(3) $\int_0^{+\infty}\frac{\sin ax\cos bx}{x}dx$
$I(a)=\int_0^{+\infty}\frac{\sin ax\cos bx}{x}dx$. $\quad\frac{\sin x}{x}\downarrow0$. $\int_0^A\cos bx\,dx$ 收敛
$I'(a)=\int_0^{+\infty}\frac{\cos bx}{x}\cdot x\cos ax\,dx=\int_0^{+\infty}\cos bx\cos ax\,dx$
$=\int_0^{+\infty}\frac{\cos bx}{a}d\sin ax$
$=\frac1a(\cos bx\sin ax\Big|_0^{+\infty}+\int_0^{+\infty}\sin ax\sin bx\cdot b\,dx)$
$\int_0^{+\infty}\frac{\sin ax\cos bx}{x}dx=\int_0^{+\infty}\frac{\sin ax}{bx}d\sin bx=\frac{\sin ax\sin bx}{bx}\Big|_0^{+\infty}-\int_0^{+\infty}\frac{\sin bx}{b}\frac{ax\cos ax-\sin ax}{x^2}dx$
$=\int_0^{+\infty}\frac{a\sin bx\cos ax\,x-\sin bx\sin ax}{bx^2}dx$

(4) $\int_0^{+\infty}\frac{\sin^2x}{x^2}dx=-\int_0^{+\infty}\sin^2x\,d\frac1x$
$=-(\sin^2x\cdot\frac1x\Big|_0^{+\infty}-\int_0^{+\infty}\frac1x2\sin x\cos x\,dx)$
$=0+\int_0^{+\infty}2\frac{\sin x}{x}\cos x\,dx$
$=\int_0^{+\infty}\frac{\sin2x}{x}dx=\int_0^{+\infty}\frac{\sin2x}{2x}d2x=\frac\pi2$.

(5) $=\int_0^{+\infty}\frac{1}{2n+1}e^{-x^2}dx^{2n+1}$ $\quad=I(n)=\int_0^{+\infty}x^{2n}e^{-x^2}dx$
$=\frac{1}{2n+1}(e^{-x^2}x^{2n+1}\Big|_0^{+\infty}-\int_0^{+\infty}x^{2n+1}e^{-x^2}\cdot(-2x)dx)$
$=\frac{1}{2n+1}\times2\int_0^{+\infty}x^{2n+2}e^{-x^2}dx$
$=\frac{2}{2n+1}\int_0^{+\infty}x^{2(n+1)}e^{-x^2}dx$
$\therefore I(n)=\frac{2}{2n+1}I(n+1)$.
$I(n+1)=\frac{2n+1}{2}I(n)=\cdots=\frac{(2n+1)!!}{2^n}I(1)$
$=\frac{(2n+1)!!}{2^{n+1}}I(0)=\frac{(2n+1)!!\sqrt\pi}{2^{n+2}}$.

(6) $\int_0^{+\infty}\frac{\sin^4x}{x^2}dx=-\int_0^{+\infty}\sin^4x\,d\frac1x$
$=-(\frac{\sin^4x}{x}\Big|_0^{+\infty}-\int_0^{+\infty}\frac{4\sin^3x\cos x}{x}dx)$
$=-\frac{\sin^4x}{x}\Big|_0^{+\infty}+\int_0^{+\infty}\frac{4(1-\cos2x)\sin2x}{x}dx$
$=\int_0^{+\infty}\frac{4(1-\cos2x)\sin2x}{2x}d2x$
$=\int_0^{+\infty}\frac{4\sin2x}{x}-\int_0^{+\infty}\frac{4\sin4x}{2x}dx=\frac\pi2-\frac\pi3$.

习题 13.5

1. (1) $\Gamma(s)=\int_0^{+\infty}t^{s-1}e^{-t}dt$
$x=\sqrt t$，$t>0$，$\Gamma(s)=\int_0^{+\infty}(x^2)^{s-1}e^{-x^2}dx^2$
$=2\int_0^{+\infty}x^{2s-2}\cdot x\,e^{-x^2}dx$.
(2) $t=ax$，$\Gamma(s)=\int_0^{+\infty}(ax)^{s-1}e^{-ax}d\,ax$
$=a^{s-1}\cdot a\int_0^{+\infty}x^{s-1}e^{-ax}dx$.

2. $B(p,q)=\int_0^1t^{p-1}(1-t)^{q-1}dt$
$=\int_0^1x^{p-1}(1-x)^{q-1}dx$.
$x=\sin^2t$，$1-x=\cos^2t$. $\quad t\in[0,\frac\pi2]$
$=\int_0^{\frac\pi2}(\sin t)^{2p-2}(\cos t)^{2q-2}d\sin^2t$
$=\int_0^{\frac\pi2}\sin^{2p-2}t\cos^{2q-2}t\times2\sin t\cos t\,dt$
$=2\int_0^{\frac\pi2}\sin^{2p-1}t\cos^{2q-1}t\,dt$

3. (1) $\int_0^1\sqrt{x(1-x)}dx$
$=\int_0^1x^{\frac12}(1-x)^{\frac12}dx$
$=\frac{\Gamma(\frac32)\Gamma(\frac32)}{\Gamma(3)}$
$=$ [illegible]

(3) $\int_0^{\frac\pi2}\sin^6x(1-\sin^2x)^2dx$
$=\int_0^{\frac\pi2}\sin^6x-2\sin^8x+\sin^{10}x\,dx$
$=\frac{5\times3\times1}{6\times4\times2}\times\frac\pi2(1-2\times\frac78+\frac{9}{10}\times\frac78)$
$=$ —.

(5) $\int_0^{+\infty}e^{-at}\frac{1}{\sqrt\pi}t^{-\frac12}dt$
$\frac{1}{\sqrt t}=\frac{2}{\sqrt\pi}\int_0^{+\infty}e^{-tu^2}du$
$=\int_0^{+\infty}e^{-at}\frac2\pi\int e^{-u^2t}du\,dt$
$=\frac2\pi\int_0^{+\infty}\int_0^{+\infty}e^{-(a+u^2)t}dt\,du$
$=\frac2\pi\int_0^{+\infty}-\frac{1}{u^2+a}e^{-(u^2+a)t}\Big|_0^{+\infty}du$
$=\frac2\pi\int_0^{+\infty}\frac{1}{u^2+a}du$
$=\frac2\pi\int_0^{+\infty}\frac{1}{(\frac{u}{\sqrt a})^2+1}\frac{du}{\sqrt a}$
$=\arctan\frac{u}{\sqrt a}\Big|_0^{+\infty}\times\frac2\pi=1$. ?

(7) $\int_0^1\sqrt{\frac1x\ln\frac1x}dx=\int_0^1x^{-\frac12}\sqrt{\ln\frac1x}dx$
$\ln\frac1x=t$，$\frac1x=e^t$，$x=e^{-t}$.
$\int_0^{+\infty}e^{\frac12t}t^{\frac12}\times(-e^{-t})dt$
$=\int_0^{+\infty}-t^{\frac12}e^{-\frac12t}dt$
$=-2\sqrt2\int_0^{+\infty}(\frac12t)^{\frac12}e^{-(\frac12t)}d\frac12t$
$=-2\sqrt2\Gamma(\frac32)=2\sqrt{2\pi}$.

(9) $\int_1^2(x-1)^2(2-x)^{\frac1n}(x-1)^{-\frac1n}dx$
$=\int_1^2(x-1)^{2-\frac1n}(2-x)^{\frac1n}dx$
$x-1=t$，$2-x=1-t$.
$=\int_0^1t^{2-\frac1n}(1-t)^{\frac1n}dt$
$=B(2-\frac1n,\frac1n)$
$=\frac{\Gamma(2-\frac1n)\Gamma(\frac1n)}{\Gamma(2)}$
$=\frac{\int_0^{+\infty}t^{1-\frac1n}e^{-t}dt\int_0^{+\infty}t^{\frac1n-1}e^{-t}dt}{\int_0^{+\infty}te^{-t}dt=2}$
$\lim_{n\to\infty}=\frac{\int_0^{+\infty}te^{-t}dt\int_0^{+\infty}t^{-1}e^{-t}dt}{2}$
$=\int_0^{+\infty}t^{-1}e^{-t}dt$
$=\int_0^{+\infty}t^{-1}de^{-t}=e^{-t}t^{-1}\Big|_0^{+\infty}$

(2) $x=a\sin t$，
$\int_0^{\frac\pi2}a^2\sin^2t\,a^2\cos^2t\,d\,a\sin t$
$=a^5\int_0^{\frac\pi2}\sin^2t(1-\sin^2t)dt$
$=a^5(\int_0^{\frac\pi2}\sin^2t-\sin^4t)dt$

(4) $x^n=t$. $x=\sqrt[n]t$.
$\int_0^1t^{\frac mn}(1-t)^{q-1}dt^{\frac1n}$
$=\frac1n\int_0^1t^{\frac{m-1}{n}+\frac1n-1}(1-t)^{q-1}dt$
$=\frac1n\int_0^1t^{\frac{m-n}{n}}(1-t)^{q-1}dt$
$=\frac1n\frac{\Gamma(\frac mn)\Gamma(q-1)}{\Gamma(\frac{m-n}{n}+q-1)}$

(6) $\int_0^{\frac\pi2}\sin^\alpha x\cos^{-\alpha}x\,dx$
$2q-1=\alpha$，$q=\frac{\alpha+1}{2}$
$2p-1=-\alpha$，$p=\frac{-\alpha+1}{2}$.
$=B(\frac{\alpha+1}{2},\frac{-\alpha+1}{2})$
$=\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{-\alpha+1}{2})}{\Gamma(1)}$

(8) $\int_a^b(b-x)^p(x-a)^{-p}dx$.
$b-x=t$. $x=b-t$. $x-a=b-a-t$
$x\in[a,b]$，$t\in[0,b-a]$.
$\int_0^{b-a}t^p(b-a-t)^{-p}d(b-t)$
$=-\int_0^{b-a}t^p(b-a-t)^{-p}dt$.
$t=(b-a)\cos^2\theta$.
$=\int_0^{\frac\pi2}(b-a)^p\cos^{2p}\theta(b-a)^{-p}\sin^{-2p}\theta$.
$\times(b-a)2\cos\theta\sin\theta\,d\theta$
$=2(b-a)\int_0^{\frac\pi2}\cos^{2p+1}\theta\sin^{-2p+1}\theta\,d\theta$

(10) $\int_0^{+\infty}\frac{1}{1+x^n}dx=\int_0^{+\infty}(x^n+1)^{-1}dx$
$t=x^n$. $x=t^{\frac1n}$. $dx=\frac1nt^{\frac1n-1}$.
$=\int_0^{+\infty}\frac{1}{(1+t)}\times\frac1nt^{\frac1n-1}dt$
$=\frac1n\int_0^{+\infty}\frac{t^{\frac{1-n}{n}}}{(1+t)}dt$ $\quad\begin{cases}q-1=\frac1n-1\\p+q=1\end{cases}$
$q=\frac1n$，$p=1-\frac1n$. $\quad$ n时无意义，但只需考虑 $n\to+\infty$.
$=\frac1nB(1-\frac1n,\frac1n)=\frac1n\frac{\Gamma(1-\frac1n)\Gamma(\frac1n)}{\Gamma(1)}=\frac1n\frac{\pi}{\sin\frac\pi n}\to1$. $\quad$ 余元公式.
$\lim_{n\to\infty}\frac1nB(1-\frac1n,\frac1n)$