补充题

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1) 三目指导

含参变量的积分

185. (1) $F(\alpha)=\int_{-1}^{1} e^{\sqrt{x^2+\alpha^2}}\,dx$　$\lim\limits_{\alpha\to 0}F(\alpha)$

$\lim\limits_{\alpha\to 0}\int_{-1}^{1} e^{\sqrt{x^2+\alpha^2}}\,dx=\int_{-1}^{1}\lim\limits_{\alpha\to 0} e^{\sqrt{x^2+\alpha^2}}\,dx$

$=\int_{-1}^{1} e^{\sqrt{x^2}}\,dx=\cancel{\left(e-\frac{1}{e}\right)}$　$2\int_0^1 e^{x}\,dx$

(2) $G(u)=\int_{\sin u}^{\cos u}\frac{1}{1+(1+u)x^2}\,dx$

$\lim\limits_{u\to 0}\int_{\sin u}^{\cos u}\frac{1}{1+(1+u)x^2}\,dx$

连续，$\int_0^1\frac{1}{1+x^2}\,dx=\frac{\pi}{4}$.

185. (1) $\int_0^{\frac{\pi}{2}}\ln(a^2\sin^2 x+b^2\cos^2 x)\,dx$

$F(u)=\int_0^{\frac{\pi}{2}}\ln(a^2\sin^2 x+u^2\cos^2 x)\,dx$

$F'(u)=\int_0^{\frac{\pi}{2}}\frac{2u\cos^2 x}{a^2\sin^2 x+u^2\cos^2 x}\,dx$

$=\int_0^{\frac{\pi}{2}}\frac{2u}{a^2\tan^2 x+u^2}\,dx$　　$\tan x=\frac{t}{au}$

$=\frac{2u}{\frac{1}{u^2}}\int_0^{\frac{\pi}{2}}\frac{1}{a^2\tan^2 x+u^2}\,dx$　　$x=\arctan\frac{t}{au}$

$=2u^3\int_0^{+\infty}\frac{1}{t^2+1}\times\frac{1}{a^2t^2+u^2}\,dt=\frac{\pi}{u+a}$.　$2u^3\arctan(au\tan x)\Big|_0^{\frac{\pi}{2}}=\pi u^3$.

$F(u)=\frac{\pi}{4}u^4$，$F=\frac{\pi}{4}b^4$

$F(b)-F(a)=\int_a^b F'(u)\,du=\pi\int_a^b\frac{1}{u+a}\,du$

186. $f(x,y)=\frac{y^2-x^2}{(x^2+y^2)^2}$

$\int_0^1 dy\int_0^1 f(x,y)\,dx\neq\int_0^1 dx\int_0^1 f(x,y)\,dy$.

$f(x,y)=\frac{y^2-x^2}{(x^2+y^2)^2}$　　x为变量 y为参数.

$\int_0^1 dy\int_0^1\frac{y^2-x^2}{(x^2+y^2)^2}\,dx=\int_0^1\left(\frac{x}{x^2+y^2}\Big|_{x=0}^{x=1}\right)dy=\int_0^1\frac{1}{1+y^2}\,dy=\frac{\pi}{4}$.

含参变量广义积分

187. (1) $\int_0^{+\infty}\alpha e^{-\alpha x}\,dx$，$0<a_0\le\alpha\le b$

$\int_0^{+\infty}\alpha e^{-\alpha x}\,dx=-\frac{1}{e^{\alpha x}}$

$(e^{-\alpha x})'=-\alpha e^{-\alpha x}$

$I(\alpha)=\int_0^{+\infty}\frac{\sin(x+\frac{1}{x})}{x^\alpha}\,dx$ 收敛性.

$=\int_0^1+\int_1^{+\infty}$,

$0\sim 1$，$y=\frac{1}{x}$. $\int_0^1\frac{\sin(x+\frac{1}{x})}{x^\alpha}\,dx=\int_1^{+\infty}\frac{\sin(y+\frac{1}{y})}{y^{2-\alpha}}\,dy$.

只需考虑 $\int_1^{+\infty}\frac{\sin(x+\frac{1}{x})}{x^\alpha}\,dx$.

$\alpha>1$，绝对收敛.　　$2-\alpha>1$

$\alpha\le 0$，发散.　　$2-\alpha\le 0$

$0<\alpha\le 1$. $\int_1^{+\infty}\frac{\sin x\cos\frac{1}{x}+\cos x\sin\frac{1}{x}}{x^\alpha}\,dx$　　$0<2-\alpha<1$.

$=\int_1^{+\infty}\cos\frac{1}{x}\frac{\sin x}{x^\alpha}+\frac{\cos x\sin\frac{1}{x}}{x^\alpha}\,dx$

$\sim\frac{\cos x}{x^{\alpha+1}}\,dx$ 条件收敛.

$\therefore 0<\alpha<2$ 条件收敛.

收敛速度. $\int_1^{+\infty}\frac{\sin x}{x^\alpha}\,dx$.

$\int_x^{+\infty}\frac{1}{t^\alpha}\,dt$.　$x^{\alpha-1}$,

$\lim\limits_{x\to+\infty}\int_1^x\frac{1}{t^\alpha}\,dt=\lim\limits_{x\to+\infty}F(x)$.

$\int_x^{+\infty}\frac{1}{t^\alpha}\,dt=\frac{1}{\alpha-1}\frac{1}{x^{\alpha-1}}$.

$\therefore x^{\alpha-1}\int_x^{+\infty}\frac{1}{t^\alpha}\,dt=\frac{1}{\alpha-1}\neq 0$.　$\lim\limits_{x\to+\infty}x^{\alpha-1}\int_x^{+\infty}\frac{1}{t^\alpha}\,dt=\frac{1}{\alpha-1}\neq 0$

$\therefore\lim\limits_{x\to+\infty}x^\beta\int_x^{+\infty}\frac{1}{t^\alpha}\,dt=\begin{cases}0, & \beta<\alpha-1\\ +\infty, & \beta>\alpha-1.\end{cases}$

$\therefore\lim\limits_{x\to+\infty}x^\beta\int_x^{+\infty}\frac{\sin t}{t^\alpha}\,dt=\begin{cases}0, & \beta<\alpha\\ +\infty, & \beta>\alpha.\end{cases}$

$x^\beta\left|\int_x^{A}\frac{\sin t}{t^\alpha}\,dt\right|=\left|\frac{1}{x^\alpha}\int_x^{\xi}\sin t\,dt\right|\le\frac{2}{x^{\alpha-\beta}}$.

$x^\alpha\int_x^{+\infty}\frac{\sin t}{t^\alpha}\,dt=x^\alpha\int_x^{+\infty}\frac{d(-\cos t)}{t^\alpha}=\cos x-\alpha x^\alpha\int_x^{+\infty}\frac{\cos t}{t^{\alpha+1}}\,dt$

化到最简（换元、迭代、分部）.
尝试写 Γ, B 函数考虑.

含参变量积分的应用.

189. $\int_0^{+\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}$，求 $\int_0^{+\infty}\frac{\sin^2 x}{x^2}\,dx$

$\int_0^{+\infty}\frac{\sin^2 x}{x^2}\,dx=-\frac{\sin^2 x}{x}\Big|_0^{+\infty}+\int_0^{+\infty}\frac{2\sin x\cos x}{x}\,dx$

$=\int_0^{+\infty}\frac{\sin 2x}{x}\,dx=\frac{\pi}{2}$.

190. (1) $\int_0^{+\infty}t^{-\frac{1}{2}}e^{-\alpha t}\,dt$

$=\int_0^{+\infty}\alpha^{\frac{1}{2}}t^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}e^{-\alpha t}\,d\alpha t\ \alpha^{-1}$

$=\alpha^{-\frac{1}{2}}\int_0^{+\infty}(\alpha t)^{-\frac{1}{2}}e^{-\alpha t}\,d\alpha t$　　$m=\alpha t$

$=\alpha^{-\frac{1}{2}}\int_0^{+\infty}m^{-\frac{1}{2}}e^{-m}\,dm$　　$A=m^{-\frac{1}{2}}$，$m=A^{-2}=\frac{1}{A^2}$

$=\alpha^{-\frac{1}{2}}\int_{+\infty}^{0}Ae^{-A^{-2}}(-2)A^{-3}\,dA$　　$A^2=m^{-1}$

$=2\alpha^{-\frac{1}{2}}\int_0^{+\infty}A^{-2}e^{-A^{-2}}\,dA$

指数 $x-1=-\frac{1}{2}$，$x=\frac{1}{2}$，
$\alpha^{-\frac{1}{2}}\Gamma(\frac{1}{2})=\sqrt{\frac{\pi}{\alpha}}$.

(2) $\int_0^{+\infty}x^{2n+1}e^{-x^2}\,dx$

$=\int_0^{+\infty}\frac{1}{(2n+2)}e^{-x^2}\,dx^{2n+2}$

$=\frac{1}{(2n+2)}\left[e^{-x^2}x^{2n+2}\Big|_0^{+\infty}-\int_0^{+\infty}-2xe^{-x^2}x^{2n+2}\,dx\right]$

$=\frac{2}{(2n+2)}\int_0^{+\infty}x^{2n+3}e^{-x^2}\,dx$.　　$x^2=t$.

$I(2n+1)=\frac{2}{(2n+2)}I(2n+3)$.　　$\frac{1}{2}\int_0^{+\infty}t^ne^{-t}\,dt=\frac{1}{2}\Gamma(n+1)=\frac{n!}{2}$

$I(2n+3)=(n+1)I(2n+1)$

$I(2n+1)=nI(2n-1)=n!\,I(1)=\frac{n!}{2}$

(3) $\int_0^{\frac{\pi}{2}}\tan^\alpha x\,dx$

$\tan x=t\in[0,+\infty)$. $x=\arctan t$.

$\int_0^{+\infty}t^\alpha\frac{1}{t^2+1}\,dt$　　$t^2=u$，$u^{\frac{1}{2}}=t$

$\int_0^{+\infty}u^{\frac{\alpha}{2}}\frac{1}{u+1}\frac{1}{2}u^{-\frac{1}{2}}\,du=\frac{1}{2}\int_0^{+\infty}\frac{u^{\frac{\alpha-1}{2}}}{u+1}\,du$

$q-1=\frac{\alpha-1}{2}$，$q=\frac{\alpha+1}{2}$. $p+q=1$，$p=\frac{1-\alpha}{2}$.

$=B(\frac{\alpha+1}{2},\frac{1-\alpha}{2})$.$=\frac{1}{2}\frac{\pi}{\sin(\frac{\pi}{2}+\frac{\alpha}{2})}=\frac{\pi}{2\cos\frac{\alpha\pi}{2}}$
$p+q=1$

(4) $\int_0^1(\ln x)^n\,dx$

$t=\ln x$，$x=e^t$.　$dx=e^t\,dt$.

$x\in(0,1)$，$t\in(-\infty,0)$.　$t\in(0,+\infty)$.

$\Rightarrow -t=\ln x$，$x=e^{-t}$.　$dx=-e^{-t}\,dt$.

$-\int_0^{+\infty}t^ne^{-t}\,dt=-\Gamma(n+1)=-n!$　$(-1)^nn!$

(5) $\int_0^{+\infty}\frac{2x^2}{(1+x^2)^3}\,dx$.

$x^2=t$. $\in[0,+\infty)$　$x=\sqrt{t}=t^{\frac{1}{2}}$. $dx=\frac{1}{2}t^{-\frac{1}{2}}\,dt$.

$\int_0^{+\infty}\frac{2t}{(1+t)^3}\frac{1}{2}t^{-\frac{1}{2}}\,dt=\int_0^{+\infty}\frac{t^{\frac{1}{2}}}{(1+t)^3}\,dt$.

$q-1=\frac{1}{2}$，$p+q=3$.　$q=\frac{3}{2}$，$p=\frac{3}{2}$.

$=B(\frac{3}{2},\frac{3}{2})=\frac{\Gamma^2(\frac{3}{2})}{\Gamma(3)}=\frac{(\frac{1}{2}\sqrt{\pi})^2}{2}=\frac{\pi}{8}$.

(6) $\int_a^b(x-a)^2\left(\frac{b-x}{x-a}\right)^p\,dx$,

$=\int_a^b(x-a)^{2-p}(b-x)^p\,dx$.

$x-a=t\in[0,b-a]$

$x=a+t$，$b-x=b-a-t$

$=\int_0^{b-a}t^{2-p}(b-a-t)^p\,dt$.

$u=\frac{t}{b-a}\in[0,1]$.　$t=(b-a)u$.

$=\int_0^1(b-a)^{2-p}u^{2-p}(b-a-(b-a)u)^p(b-a)\,du$　$(b-a)^p(1-u)^p$

$=(b-a)^3\int_0^1u^{2-p}(1-u)^p\,du$.

$=(b-a)^3B(3-p,1+p)$.

$=(b-a)^3\frac{\Gamma(3-p)\Gamma(1+p)}{\Gamma(4)}$

$=\frac{(b-a)^3p(2-p)(1-p)}{6}$　$\frac{\pi}{\sin p\pi}$.

191. (1) $\lim\limits_{n\to+\infty}\int_0^{+\infty}\frac{1+x^n}{1+x^{2n}}\,dx$.

$x^n=t$，$x=t^{\frac{1}{n}}$.

$\lim\limits_{n\to+\infty}\int_0^{+\infty}\frac{1+t}{1+t^2}\frac{1}{n}t^{\frac{1}{n}-1}\,dt$.

$=\lim\limits_{n\to+\infty}\frac{1}{n}\int_0^{+\infty}\frac{1+t}{1+t^2}t^{\frac{1}{n}-1}\,dt$.

$=\lim\limits_{n\to+\infty}\frac{1}{n}\left(\int_0^{+\infty}\frac{t^{\frac{1}{n}-1}}{1+t^2}\,dt+\int_0^{+\infty}\frac{t^{\frac{1}{n}}}{1+t^2}\,dt\right)$

$\frac{1}{2n}-1=q-1$　　应该 $x^{2n}=t$.

$p=1-\frac{1}{2n}$　$=\lim\limits_{n\to+\infty}\frac{1}{2n}\left(\int_0^{+\infty}\frac{t^{\frac{1}{2n}-1}}{1+t}\,dt+\int_0^{+\infty}\frac{t^{\frac{1}{2n}-\frac{1}{2}}}{1+t}\,dt\right)$

$=\lim\limits_{n\to+\infty}\frac{1}{2n}\left(B(1-\frac{1}{2n},\frac{1}{2n})+B(\frac{1}{2}-\frac{1}{2n},\frac{1}{2n}+\frac{1}{2})\right)$

$=\lim\limits_{n\to+\infty}\frac{1}{2n}\left(\Gamma(1-\frac{1}{2n})\Gamma(\frac{1}{2n})+\Gamma(\frac{1}{2}-\frac{1}{2n})\Gamma(\frac{1}{2}+\frac{1}{2n})\right)$

$=\lim\limits_{n\to+\infty}\frac{1}{2n}\left(\frac{\pi}{\sin\frac{\pi}{2n}}+\frac{\pi}{\sin(\frac{1}{2}+\frac{1}{2n})\pi}\right)$

$=\lim\limits_{n\to+\infty}\left(\frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}+\frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\right)=1$.

$\downarrow$ 1　　$\to 0$

(2) $\lim\limits_{n\to\infty}\int_0^{+\infty}x^2e^{-x^{2n}}\,dx$

$x^{2n}=t$，$x=t^{\frac{1}{2n}}$.

$\lim\limits_{n\to+\infty}\int_0^{+\infty}t^{\frac{1}{n}}e^{-t}\frac{1}{2n}t^{\frac{1}{2n}-1}\,dt$

$=\lim\limits_{n\to+\infty}\int_0^{+\infty}t^{\frac{3}{2n}-1}e^{-t}\,dt\ \frac{1}{2n}$.

$=\lim\limits_{n\to+\infty}\frac{\Gamma(\frac{3}{2n})}{2n}$　　$=\lim\limits_{n\to\infty}\frac{1}{3}\Gamma(\frac{3}{2n}+1)$
$=\frac{1}{3}\Gamma(1)=\frac{1}{3}$.