补充题

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$\int_0^{\pi} \cos nx\,dx =$

$\int_0^{\pi} \sin nx\,dx =$

《学习指导》

192. $f(x)=x^2,\ x\in[0,2]$.

(1) 以2为周期：$a_0=\frac{1}{\frac{2}{2}}\int_0^2 x^2dx$

$\int_0^{\pi} x^2\cos 2nx\,dx=\frac{\pi}{2n^2}$ $\quad a_n=\frac{1}{\frac{2}{2}}\int_0^2 x^2\cos\frac{n2\pi x}{2}dx$

$\int u^2\cos u\,du=\int u^2 d\sin u$ $\quad =\frac{2}{\pi}\int_0^{\pi} x^2\cos 2nx\,dx=\frac{1}{n^2}$.

$=u^2\sin u-2\int u\sin u\,du$ $\quad =\frac{2}{\pi}\int_0^{\pi} n^2x^2\cos 2nx\cdot\frac{1}{n^3}\,dnx$

$=u^2\sin u+2\int u\,d\cos u$ $\quad =\frac{2}{\pi}\int_0^{n\pi} t^2\cos 2t\,\frac{1}{n^3}dt$

$=u^2\sin u+2u\cos u-2\int\cos u\,du$ $\quad =\frac{1}{4\pi}\int_0^{2n\pi} u^2\cos u\,\frac{1}{n^3}du$

$=u^2\sin u+2u\cos u-2\sin u$ $\quad =\frac{1}{4\pi n^3}\times 2\cdot 2n\pi\times 1=\frac{1}{n^2}$.

$b_n=\frac{1}{\frac{2}{2}}\int_0^2 x^2\sin\frac{n2\pi x}{2}dx$

$=\frac{2}{\pi}\int_0^{\pi} x^2\sin 2nx\,dx$

$=\frac{1}{4\pi n^3}\int_0^{2n\pi} u^2\sin u\,du$

$=\cdots$

(2) 以 2π 为周期余

$f(x)=x^2,\ x\in[0,\pi]$.

$a_0=\frac{2}{\pi}\int_0^{\pi} x^2dx$

$a_n=\frac{2}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx=\frac{4(-1)^n}{n^2}$.

(3) 以 2π 为周期正.

$f(x)=\begin{cases} x^2 \\ -x^2 \end{cases}$.

$b_n=\frac{2}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$

193. $f(x)=\begin{cases}0 & 1\le x<2\\ x-2 & 2\le x<3\end{cases}$. 以2为周期.

$a_0=\frac{1}{\frac{2}{2}}\int_1^3 f(x)dx$

$=\int_1^2 0\,dx+\int_2^3 x-2\,dx=\frac{1}{2}x^2-2x\Big|_2^3=\frac{9}{2}-6-\frac{4}{2}+4=\frac{1}{2}$.

$a_n=\frac{1}{\frac{2}{2}}\int_1^3 f(x)\cos\frac{n2\pi}{2}x\,dx$

$=\int_1^2 0\cos n\pi x\,dx+\int_2^3(x-2)\cos n\pi x\,dx$

$=\int_2^3 x\cos n\pi x\,dx-2\int_2^3\cos n\pi x\,dx$

$=\frac{1}{n^2\pi^2}\int_2^3 n\pi x\cos n\pi x\,dn\pi x-\frac{2}{n\pi}\int_2^3\cos n\pi x\,dn\pi x$

$=\frac{1}{n^2\pi^2}(t\sin t+\cos t)\Big|_{2n\pi}^{3n\pi}-\frac{2}{n\pi}\times 0$

$=\frac{1}{n^2\pi^2}(\cos 3n\pi-\cos 2n\pi)=\frac{(-1)^n-1}{n^2\pi^2}$

$b_n=\int_1^2 0\sin n\pi x+\int_2^3 x\sin n\pi x-2\sin n\pi x\,dx$

$=\frac{1}{n^2\pi^2}\int_2^3 n\pi x\sin n\pi x\,dn\pi x-2\frac{1}{n\pi}\int_2^3\sin n\pi x\,dn\pi x$

$=\frac{1}{n^2\pi^2}(\sin t-t\cos t)\Big|_{2n\pi}^{3n\pi}+\frac{2}{n\pi}\cos t\Big|_{2n\pi}^{3n\pi}$

$=\frac{3n\pi(-1)^n-2n\pi}{n^2\pi^2}+\frac{2}{n\pi}((-1)^n-1)$

$=\frac{3(-1)^n-2+2(-1)^n-2}{n\pi}=\frac{(-1)^{n-1}}{n\pi}$?

$f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2\pi^2}\cos n\pi x+\frac{(-1)^{n-1}}{n\pi}\sin n\pi x\right]$.

$x=2$. $\quad \frac{1}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi^2}=0$. $\quad n$为奇，$\frac{1}{4}-\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}=0$.

即 $\sum\frac{1}{(2n-1)^2}\cdot\frac{2}{\pi^2}=\frac{1}{4}$.

$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{l}\int_{-l}^{l}f^2(x)\,dx$.

$\frac{1}{2}\times\frac{1}{4}+\sum_{n=1}^{\infty}\left(\frac{2^2}{(2n-1)^4\pi^4}+\frac{1}{n^2\pi^2}\right)=\int_2^3(x-2)^2dx$

194. $f(x)=\begin{cases}1, & x\in(0,2)\\ 0, & x=0\\ -1, & x\in(-2,0)\end{cases}$

$b_n=\frac{1}{2}\int_{-2}^{2}f(x)\sin nx\,dx=\frac{1}{2}\int_{-2}^{0}-\sin nx\,dx+\frac{1}{2}\int_0^2\sin nx\,dx$

$=\frac{1}{n\pi}\int_0^2\sin x\,dnx=\frac{2}{n\pi}(-\cos t)\Big|_0^{n\pi}=\frac{2(1-(-1)^n)}{n\pi}=\frac{4}{(2n-1)\pi}$.

$S_1=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\cdots$

$=\sum_{n=0}^{\infty}\frac{1}{4n+1}-\sum_{n=0}^{\infty}\frac{1}{4n+3}$.

$f(x)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}$

令 $x=\frac{\pi}{2}$，$S_1=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)}=\frac{\pi}{4}$.

$\frac{1}{2}\int_{-2}^{2}f^2(x)dx=\frac{1}{2}\int_{-2}^{2}1\,dx=2=\sum\frac{16}{(2n-1)^2\pi^2}$

$\sum\frac{1}{(2n-1)^2}=\frac{\pi^2}{16}\times 2$

$\int_0^{\pi} x\cos nx\,dx=\frac{(-1)^n-1}{n^2}$

$\int_0^{\pi} x\sin nx\,dx=\frac{(-1)^{n+1}}{n\pi}$

$\int_0^{\pi} x^2\cos nx\,dx=[u^2\sin u+2u\cos u-2\sin u]\frac{1}{n^3}\Big|_0^{n\pi}$

$\int_0^{\pi} x^2\sin nx\,dx=$

$\int x^2\cos x\,dx=x^2\sin x+2x\cos x-2\sin x$

$\int x^2\sin x\,dx=$

198. $f_p(x)=\begin{cases}\frac{1}{2p}, & |x|\le p\\ 0, & |x|>p\end{cases}$

$F_p(\lambda)=2\int_0^{+\infty}f_p(x)\cos\lambda x\,dx=2\int_0^{p}\frac{1}{2p}\cos\lambda x\,dx=\begin{cases}\frac{\sin p\lambda}{p\lambda}, & \lambda\ne 0\\ 1, & \lambda=0.\end{cases}$

$\frac{1}{\pi}\int_0^{+\infty}F_p(\lambda)\cos\lambda x\,d\lambda=\frac{1}{\pi}\int_0^{+\infty}\frac{\sin p\lambda}{p\lambda}\cos\lambda x\,d\lambda=\begin{cases}f_p(x), & |x|<p\\ \frac{1}{4p}, & |x|=p\\ 0, & |x|>p.\end{cases}$

199. $\int_0^{+\infty}g(t)\cos xt\,dt=\begin{cases}\cos x, & |x|\le\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}$

$\frac{1}{\pi}\int_0^{+\infty}g(\lambda)\cos\lambda x\,d\lambda=\begin{cases}\frac{1}{\pi}\cos x\\ 0\end{cases}$ ~~$g(\lambda)$~~ $=f(x)$.

$g(x)=\begin{cases}\frac{1}{\pi}\cos x, & x\le\frac{\pi}{2}\\ 0, & x>\frac{\pi}{2}\end{cases}$ $\quad g(t)$是$f(x)$的余弦变换.

$g(t)=2\int_0^{+\infty}f(x)\cos\lambda x\,dx=\frac{2\cos\frac{\pi}{2}\lambda}{1-\lambda^2}$

200. $f(x)=x^{-a}$, $\quad -c=u-1$.

$F(\lambda)=2\int_0^{+\infty}x^{-a}\sin\lambda x\,dx=\Gamma(1-a)\sin\frac{2(1-a)}{2}$

$f(x)=\frac{1}{\pi}\int_0^{+\infty}\Gamma(1-a)\sin\frac{\pi(1-c)}{2}$

$\int_0^{2\pi} e^x\cos nx\,dx=$

$\int_0^{2\pi} e^x\sin nx\,dx=$

$\int_0^{\frac{2\pi}{\omega}}\sin\omega t\cos n\omega t\,dt=\frac{\omega}{2}\int_0^{\frac{2\pi}{\omega}}\sin(n+1)\omega t-\sin(n-1)\omega t\,dt$

$=\frac{(-1)^{n+1}-1}{(n^2-1)}$

$\int_0^{\frac{2\pi}{\omega}}\cos\omega t\sin n\omega t\,dt=\frac{\omega}{2}\int_0^{\frac{2\pi}{\omega}}\cos(n-1)\omega t-\cos(n+1)\omega t\,dt=0$

1. 引入.

L^p space $\quad \|f\|_p=(\int_a^b|f(x)|^p dx)^{\frac{1}{p}}$.

线性空间，$\|\cdot\|_p$ 是一个范数（内积性质）

$C^{\infty}([a,b])$ 在 $L^p(a,b],\|\cdot\|_p$ 中稠密.

$p=2$ 时. $L^2[a,b],\|\cdot\|_2$ 是内积空间.

寻找一个空间的稠密子集 D，

D中元素尽可能少.

$\Rightarrow$ 取 $[a,b]=[-\pi,\pi]$，

$B=\{\sin nx\}_{n\ge1}\cup\{\cos nx\}_{n\ge0}$，$T=\operatorname{span}B$.

则 T 在 $L^2[-\pi,\pi]$ 中稠密.

2. 基.

欧氏空间中取一组基可以进行标准正交化.

使 $\forall x=\sum_{i=1}^{n}a_ie_i$，

且 $|x|^2=\sum(a_i)^2$，$(x,y)=\sum a_ib_i$.

则 B 中的元素基本满足.

$(1,1)=2\pi$，$(1,\cos nx)=(1,\sin nx)=0$

$(\sin mx,\sin nx)=(\cos mx,\cos nx)=\pi\delta_{mn}$.

$\Rightarrow$ 标准正交基 $\{\frac{\sin nx}{\sqrt{\pi}}\}_{n\ge1}\cup\{\frac{\cos nx}{\sqrt{\pi}}\}_{n\ge1}\cup\{\frac{1}{\sqrt{2\pi}}\}$.

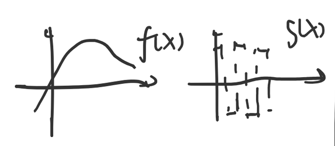
$\therefore f\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$=(f,\frac{1}{\sqrt{2\pi}})\frac{1}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}(f,\frac{\sin nx}{\sqrt{\pi}})\frac{\sin nx}{\sqrt{\pi}}+(f,\frac{\cos nx}{\sqrt{\pi}})\frac{\cos nx}{\sqrt{\pi}})$

由此可验证 Parseval 等.

3. 部分和.

卷积的特殊性质 → 调整连续性.

黎曼引理的几何意义 → 

$f(x)$受$g(x)$的权重影响.

$S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\,dy+\frac{1}{\pi}\sum_{k=1}^{n}\int_{-\pi}^{\pi}f(y)\cos ky\,dy\cos kx+\int_{-\pi}^{\pi}f(y)\sin ky\,dy\sin kx$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy$

$=\frac{1}{2\pi}\left(\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}*f\right)(x)$ （卷积）

由Riemann引理，$n\to+\infty$，$\sin S_n(x)\to0$.

取部分区间.（调整区间）

用灵活的函数拟合近

Gibbs 现象.