例题

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$\int_0^{\pi} \cos nx\,dx = \int_0^{\pi} \frac{1}{n}\cos nx\,dnx$

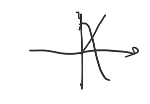
$= \frac{1}{n}\sin nx\Big|_0^{\pi}$

$= \frac{1}{n}\sin n\pi = 0$

$\int_0^{\pi} \sin nx\,dx = \int_0^{\pi}\frac{1}{n}\sin nx\,dnx$

$= -\frac{1}{n}\cos nx\Big|_0^{\pi}$

$= \frac{1-(-1)^n}{n}$



$\int_0^{\pi} x\cos nx\,dx = \int_0^{\pi}\frac{1}{n^2} nx\cos nx\,dnx$

$= \frac{1}{n^2}\int_0^{\pi} nx\,d\sin nx$

$= \frac{1}{n^2}\left(nx\sin nx\Big|_0^{\pi} - \int_0^{\pi}\sin nx\,dnx\right)$

$= \frac{1}{n^2}(-1+(-1)^n) = \frac{(-1)^n-1}{n^2}$

$\int_0^{\pi} x\sin nx\,dx = \int_0^{\pi}\frac{1}{n^2} nx\sin nx\,dnx$

$= \frac{1}{n^2}\int_0^{\pi} -nx\,d\cos nx$

$= -\frac{1}{n^2}\left(nx\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\,dnx\right)$

$= -\frac{1}{n^2}((-1)^n n\pi - 0) = \frac{(-1)^{n+1}\pi}{n}$

$\int_0^{\pi} x^2\cos nx\,dx = \int_0^{\pi}\frac{1}{n^3}(nx)^2\cos nx\,dnx$

$= \frac{1}{n^3}\left((nx)^2\sin nx\Big|_0^{\pi} - \int_0^{\pi}\sin nx\,d(nx)^2\right)$

$= \frac{1}{n^3}\left(-2\int_0^{\pi}(nx)\sin nx\,dnx\right)$

$= \frac{-2}{n^3}\left(-\int_0^{\pi} nx\,d\cos nx\right)$

$= \frac{2}{n^3}\left(nx\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\,dnx\right)$

$= \frac{2}{n^3}(\pi(-1)^n - 0) = \frac{(-1)^n 2\pi}{n^2}$

$\int_0^{\pi} x^2\sin nx\,dx = \int_0^{\pi}\frac{1}{n^3}(nx)^2\sin nx\,dnx$

$= -\frac{1}{n^3}\left[(nx)^2\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\,d(nx)^2\right]$

$= -\frac{1}{n^3}\left(n^2\pi^2(-1)^n - 2\int_0^{\pi}(nx)\cos nx\,dnx\right)$

$= -\frac{1}{n}\left(\pi^2(-1)^n - 2\int_0^{\pi}x\cos nx\,dx\right)$

$= -\frac{1}{n}\left(\pi^2(-1)^n - \frac{2((-1)^n-1)}{n^2}\right)$

$= \frac{2[(-1)^n-1] - n^2\pi^2(-1)^n}{n^3}$

$\int t\cos t\,dt = \int t\,d\sin t$

$= t\sin t - \int\sin t\,dt$

$= t\sin t + \cos t$

$\int t\sin t\,dt = -\int t\,d\cos t$

$= -t\cos t + \int\cos t\,dt$

$= -t\cos t + \sin t$

$\int t^2\cos t\,dt = t^2\sin t - 2\int t\sin t\,dt$

$= t^2\sin t + 2\int t\,d\cos t$

$= t^2\sin t + 2t\cos t - 2\int\cos t\,dt$

$= t^2\sin t + 2t\cos t - 2\sin t$

$\int t^2\sin t\,dt = -t^2\cos t + 2\int t\cos t\,dt$

$= -t^2\cos t + 2\int t\,d\sin t$

$= -t^2\cos t + 2t\sin t + 2\int\sin t\,dt$

$= -t^2\cos t + 2t\sin t + 2\cos t\Big|_0^{\pi}$

$\frac{(-1)^n n^2\pi^2 - 2(-1)^n}{n^3}$

$\int e^t\cos t\,dt = \int\cos t\,de^t$

$= e^t\cos t + \int e^t\sin t\,dt$

$= e^t\cos t + \int\sin t\,de^t$

$= e^t\cos t + e^t\sin t - \int\cos t\,e^t\,dt$

$= \frac{e^t}{2}(\cos t + \sin t)$

$\int e^t\sin t\,dt = e^t\sin t - \int e^t\cos t\,dt$

$= e^t\sin t - e^t\cos t - \int e^t\sin t\,dt$

$= \frac{e^t}{2}(\sin t - \cos t)$

一. Fourier 级数

例. 设 $f(x)$ 以 2π 为周期，$f(x)=\begin{cases} x, & 0\le x<\pi \\ 0, & -\pi\le x<0.\end{cases}$

展成 Fourier 级数.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\cdot\frac{1}{2}\pi^2 = \frac{\pi}{2}.$

$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$

$= \frac{1}{\pi}\frac{1}{n^2}\int_0^{\pi} nx\cos nx\,dnx$

$= \frac{1}{\pi}\frac{1}{n^2}\int_0^{n\pi} t\cos t\,dt$

$= \frac{1}{n^2\pi}(t\sin t + \cos t)\Big|_0^{n\pi} = \frac{(-1)^n-1}{n^2\pi}$

$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$

$= \frac{1}{n^2\pi}\int_0^{n\pi} t\sin t\,dt$

$= \frac{1}{n^2\pi}(-t\cos t + \sin t)\Big|_0^{n\pi} = \frac{-n\pi(-1)^n}{n^2\pi} = \frac{(-1)^{n+1}}{n}$

$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n^2\pi}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right)$

$= \begin{cases} f(x), & x\ne(2k-1)\pi \\ \frac{\pi}{2}, & x=(2k-1)\pi.\end{cases}$

$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{-2\cos nx}{(2n-1)^2\pi} + \frac{1}{(2n-1)}\sin(2n-1)x\right)$

$x=0$，$0 = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$，$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$

$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4}{3}\times\frac{\pi^2}{8} = \frac{\pi^2}{6}.$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2}$，$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$

例. $f(x)=\cos ax$ 在 $[-\pi,\pi]$ 上的 Fourier 级数.

其中 a 不是整数.

偶函数 ⇒ 仅有 a_n.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos ax\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi}\cos(a-n)x + \cos(a+n)x\,dx$

$= \frac{1}{\pi(a-n)}\sin t\Big|_0^{(a-n)\pi} + \frac{1}{\pi(a+n)}\sin t\Big|_0^{(a+n)\pi}$

$= \frac{1}{\pi(a-n)}(-1)^n\sin a\pi + \frac{1}{\pi(a+n)}(-1)^n\sin a\pi$

$= \frac{(-1)^n(a+n)\sin a\pi + (-1)^n(a-n)\sin a\pi}{\pi(a^2-n^2)}$

$= \frac{(-1)^n 2a\sin a\pi}{\pi(a^2-n^2)}$

$\therefore \cos ax = \frac{\sin a\pi}{\pi}\left(\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}\cos nx\right)$

$\frac{\pi}{\sin a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}.$

1. $f(x)=\begin{cases} e^x & -\pi\le x<0 \\ 1 & 0\le x<\pi.\end{cases}$

$\cos nx = \frac{e^{inx}+e^{-inx}}{2}$

$\frac{1}{2}\left[e^{x(1+in)} + e^{x(1-in)}\right]$

$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^0 e^x\,dx + \int_0^{\pi}1\,dx\right)$

$= \frac{1}{\pi}(1 - e^{-\pi} + \pi)$

$a_n = \frac{1}{\pi}\left[\int_{-\pi}^0 e^x\cos nx\,dx + \int_0^{\pi}\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-n\pi}^0 \frac{1}{n}e^{\frac{t}{n}}\cos t\,dt + \frac{1}{n}\sin t\Big|_0^{n\pi}\right]$

$= \frac{1}{\pi n e^{?}}\frac{e^{t}}{2}(\cos t + \sin t)\Big|_{-n\pi}^0 + \frac{1}{\pi}\frac{1}{n}\sin t\Big|_0^{n\pi}$

$= \frac{1-(-1)^n e^{-n\pi}}{\pi n 2 e^{?}}$

$b_n = \frac{1}{\pi}\left(\int_{-\pi}^0 e^x\sin nx\,dx + \int_0^{\pi}\sin nx\,dx\right)$

$= \frac{1}{\pi}\left(\int_{-n\pi}^0 \frac{1}{n e^{?}}e^{t}\sin t\,dt - \frac{1}{n}\cos nx\Big|_0^{\pi}\right)$

$= \frac{1}{2\pi n e^{?}}\frac{e^t}{2}(\sin t - \cos t)\Big|_{-n\pi}^0 - \frac{1}{\pi n}\cos t\Big|_0^{n\pi}$

$= \frac{1-(-1)^n e^{-n\pi}}{2\pi n e^{?}}$

$= \frac{1-(-1)^n e^{-n\pi} - 2(-1)^n e^{?} + 2\pi n e^{?}}{2\pi n e^{?}}$

例. (综合T1). 证明：$\sum\frac{\sin nx}{n}$ 不只包含 2π 整数倍的闭区间上一致收敛，但它不是任意在 $[-\pi,\pi]$ 上平方可积函数的 Fourier 级数.

当 $x\in[a,b]$ 时，$0<a<b<2\pi$（不包含 2π）.

$\sum_{k=1}^{n}\sin kx = \sum_{k=1}^{n}\frac{\sin\frac{x}{2}\sin kx}{\sin\frac{x}{2}} = \sum_{k=1}^{n}\frac{\cos(k-\frac{1}{2})x - \cos(k+\frac{1}{2})x}{2\sin\frac{x}{2}}$

$= \frac{\cos\frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$

$\therefore\left|\sum_{k=1}^{n}\sin kx\right| \le \frac{1}{\sin\frac{a}{2}} \le M$，该级数一致收敛.（Dirichlet）

一致收敛

又 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 发散，根据 Parseval，$f(x)$ 不平方可积.

例. (综合T2) f 是周期 2π 的可积且绝对可积函数，若 $f(x)$ 在 $(0,2\pi)$ 递减，则 $b_n\ge 0$.

只证 $f(x)$ 在 $[0,2\pi]$ 有界时.

$\pi n\cdot b_n = n\int_0^{2\pi} f(x)\sin nx\,dx = \int_0^{2\pi} f\left(\frac{t}{n}\right)\sin t\,dt$（积分第二中值定理）

$= \int_0^{2n\pi} f\left(\frac{t}{n}\right)\sin t\,dt$（注明中间步骤）

$\sum_{k=0}^{n-1}\left[\int_{2k\pi}^{(2k+1)\pi} f\left(\frac{t}{n}\right)\sin t\,dt + \int_{(2k+1)\pi}^{(2k+2)\pi} f\left(\frac{t}{n}\right)\sin t\,dt\right]$

$= \sum_{k=0}^{n-1}\left[\int_0^{\pi} f\left(\frac{u+2k\pi}{n}\right)\sin u\,du + \int_0^{\pi} f\left(\frac{u+(2k+1)\pi}{n}\right)(-\sin u)\,du\right]$

$= \sum_{k=0}^{n-1}\int_0^{\pi}\left[f\left(\frac{u+2k\pi}{n}\right) - f\left(\frac{u+(2k+1)\pi}{n}\right)\right]\sin u\,du \ge 0.$

例. (综合T5) f 在 $[-a,a]$ 上连续，在 $x=0$ 可导.

求证：$\lim_{\lambda\to+\infty}\int_{-a}^{a}\frac{1-\cos\lambda x}{x}f(x)\,dx = \int_0^a\frac{f(x)-f(-x)}{x}\,dx.$

$\int_{-a}^{a}\frac{1-\cos\lambda x}{x}f(x)\,dx$

$= \int_0^a\frac{1-\cos\lambda x}{x}f(x)\,dx + \int_{-a}^0\frac{1-\cos\lambda x}{x}f(x)\,dx$

$= \int_0^a\frac{1-\cos\lambda x}{x}f(x)\,dx - \int_0^a\frac{1-\cos\lambda x}{x}f(-x)\,dx$

$= \int_0^a(1-\cos\lambda x)\frac{f(x)-f(-x)}{x}\,dx$

$= \int_0^a\frac{f(x)-f(-x)}{x}\,dx - \int_0^a\cos\lambda x\frac{f(x)-f(-x)}{x}\,dx.$

令 $F(x)=\begin{cases}\frac{f(x)-f(-x)}{x}, & 0<x\le a \\ 2f'(0), & x=0\end{cases}$　$\lim_{x\to0^+}\frac{f(x)-f(0)}{x} + \frac{f(0)-f(-x)}{-x} = f'(0) + f'(0)$

$\therefore F(x)$ 在 $[0,a]$ 上连续.

* Reimann-Lebesgue 引理：

设 $f(x)$ 在 $[a,b]$ 上黎曼可积，函数 $g(x)$ 以 T 为周期，且 $g(x)$ 在 $[0,T]$ 上可积，则

$\lim_{n\to+\infty}\int_a^b f(x)g(nx)\,dx = \frac{1}{T}\int_0^T g(x)\,dx\int_a^b f(x)\,dx.$

广义：$\lim_{\lambda\to+\infty}\int_a^b f(x)g(\lambda x)\,dx = \frac{1}{T}\int_0^T g(x)\,dx\int_a^b f(x)\,dx$

推论：① $\lim_{\lambda\to+\infty}\int_a^b f(x)\cos\lambda x\,dx = 0$（$\int_0^{2\pi}\cos\lambda x/\sin\lambda x\,dx = 0$）.

$\because F(x)$ 在 $[0,a]$ Reimann 可积，

$\therefore \int_0^a\frac{f(x)-f(-x)}{x}\cos\lambda x\,dx \to 0$

② $\lim_{\lambda\to+\infty}\int_a^b f(x)|\sin\lambda x|\,dx = \frac{1}{\pi}\int_0^{\pi}|\sin\lambda x|\,dx\int_a^b f(x)\,dx$

$= \frac{2}{\pi}\int_a^b f(x)\,dx.$

证明：先证 $\lambda=n$（正整数）的情形.

(1) 先设 $g(x)\ge 0$.

取正整数 m，使 $[a,b]\subset[-mT,mT]$.

令 $F(x)=\begin{cases} f(x), & x\in[a,b] \\ 0, & x\in[-mT,mT]\setminus[a,b].\end{cases}$

$\int_a^b f(x)g(nx)\,dx = \int_{-mT}^{mT}F(x)g(nx)\,dx$

$= \sum_{k=0}^{2mn-1}\int_{-mT+\frac{kT}{n}}^{-mT+\frac{(k+1)T}{n}}F(x)g(nx)\,dx = \sum_{k=0}^{2mn-1}\Delta_k.$

当 $0\le k\le 2mn-1$ 时，记 $I_k=[-mT+\frac{kT}{n}, -mT+\frac{(k+1)T}{n}]$.

记 $m_k=\inf F(x)$，$M_k=\sup F(x)$，$x\in I_k$.

易得 $m_k\int_{I_k}g(nx)\,dx \le \Delta_k \le M_k\int_{I_k}g(nx)\,dx.$

$\therefore\exists\,\mu_k\in[m_k,M_k]$ 使 $\Delta_k = \mu_k\int g(nx)\,dx$

$\int_{I_k}g(nx)\,dx = \frac{1}{n}\int_{-mnT+kT}^{-mnT+(k+1)T}g(t)\,dt = \frac{1}{n}\int_0^T g(x)\,dx$

$\therefore\int_a^b f(x)g(nx)\,dx = \sum_{k=0}^{2mn-1}\mu_k\frac{T}{n}\cdot\frac{1}{T}\int_0^T g(x)\,dx.$

$\therefore\lim_{n\to+\infty}\int_a^b f(x)g(nx)\,dx = \int_{-mT}^{mT}F(x)\,dx\cdot\frac{1}{T}\int_0^T g(x)\,dx$

$= \int_a^b f(x)\,dx\cdot\frac{1}{T}\int_0^T g(x)\,dx.$

(2) $g(x)=g_1(x)-g_2(x)$，$g_1(x)=\frac{|g(x)|+g(x)}{2}$，

$g_2(x)=\frac{|g(x)|-g(x)}{2}$

$\therefore\lim_{n\to\infty}\int_a^b f(x)g(nx)\,dx = \lim_{n\to\infty}\int_a^b f(x)g_1(nx) - f(x)g_2(nx)\,dx$

$= \int_a^b f(x)\,dx\,\frac{1}{T}\int_0^T g_1(x)-g_2(x)\,dx.$

$= \int_a^b f(x)\,dx\,\frac{1}{T}\int_0^T g(x)\,dx.$

再证广义情形.

仅需说明 $g(x)\ge0$ 部分.

记 $\int_a^b f(x)\,dx = F$，$\frac{1}{T}\int_0^T g(x)\,dx = G$.

对 $\forall\varepsilon>0$，存在分割 T 使

$F-\varepsilon < \sum_{k=1}^{n}m_k\Delta x_k$，$\sum_{k=1}^{n}M_k\Delta x_k < F+\varepsilon$.

$\therefore\sum_{k=1}^{n}m_k\int_{x_{k-1}}^{x_k}g(\lambda x)\,dx \le \int_a^b f(x)g(\lambda x)\,dx$

引理 $\lim_{\lambda\to\infty}\int_a^b g(\lambda x)\,dx = \frac{b-a}{T}\int_0^T g(x)\,dx$

$= \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}f(x)g(\lambda x)\,dx \le \sum_{k=1}^{n}M_k\int_{x_{k-1}}^{x_k}g(\lambda x)\,dx$

又 $\lim_{\lambda\to+\infty}\sum_{k=1}^{n}m_k\int_{x_{k-1}}^{x_k}g(\lambda x)\,dx = \sum_{k=1}^{n}m_k\Delta x_k\cdot\frac{1}{T}\int_0^T g(x)\,dx.$

$= \sum_{k=1}^{n}m_k\Delta x_k\,G > (F-\varepsilon)G$

$\lim_{\lambda\to+\infty}\sum_{k=1}^{n}M_k\int_{x_{k-1}}^{x_k}g(\lambda x)\,dx = \sum_{k=1}^{n}M_k\Delta x_k\cdot\frac{1}{T}\int_0^T g(x)\,dx < (F+\varepsilon)G$

$\therefore$ 存在 $\lambda_0>0$，使当 $\lambda>\lambda_0$ 时，

$\sum_{k=1}^{n}M_k\int_{x_{k-1}}^{x_k}g(\lambda x)\,dx < \sum_{k=1}^{n}M_k\Delta x_k\,G+\varepsilon$

$\sum_{k=1}^{n}m_k\int_{x_{k-1}}^{x_k}g(\lambda x)\,dx > \sum_{k=1}^{n}m_k\Delta x_k\,G-\varepsilon.$

$\therefore(F-\varepsilon)G-\varepsilon \le \int_a^b f(x)g(\lambda x)\,dx \le (F+\varepsilon)G+\varepsilon.$

$\therefore -(1+G)\varepsilon < \int_a^b f(x)g(\lambda x)\,dx - FG < (1+G)\varepsilon.$

$\therefore\lim_{\lambda\to\infty}\int_a^b f(x)g(\lambda x)\,dx = FG.$

只需证 $\lim_{\lambda\to\infty}\int_0^b g(\lambda x)\,dx = bG$，$a=0$.

当 $\lambda>\frac{T}{b}$ 时，设 $\lambda b = nT+r$，$n\in N^*$，$0\le r<T$，$n=[\frac{\lambda b}{T}]$.

$\therefore\int_0^b g(\lambda x)\,dx = \frac{1}{\lambda}\int_0^{\lambda b}g(x)\,dx = \frac{1}{\lambda}\left[\int_0^{nT}g(x)\,dx + \int_{nT}^{nT+r}g(x)\,dx\right]$

$= \frac{1}{\lambda}\left[n\int_0^T g(x)\,dx + \int_0^r g(x)\,dx\right]$

$= \frac{bnT}{\lambda b}\cdot\frac{1}{T}\int_0^T g(x)\,dx + \frac{1}{\lambda}\int_0^r g(x)\,dx$

$\to b\cdot\frac{1}{T}\int_0^T g(x)\,dx + 0$

其中 $\lim_{\lambda\to+\infty}\frac{nT}{\lambda b} = \lim_{\lambda\to+\infty}\frac{[\frac{\lambda b}{T}]}{\frac{\lambda b}{T}} = 1$，$(b>0)$.

$\left|\frac{1}{\lambda}\int_0^r g(x)\,dx\right| \le \frac{MT}{\lambda}.$

若 $b<0$，设 $-b=b_1>0$，

$\int_0^b g(\lambda x)\,dx = \frac{1}{\lambda}\int_0^{\lambda b}g(x)\,dx$

$= \lambda\int_0^{\lambda b_1}g_1(t)\,dt \to -bG \to b_1G.$

例. (综合T7). f 周期 2π 连续，令

$F(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt$，$a_n, b_n$ 和 A_n, B_n

表示 f, F 的 Fourier 系数，证明：

$A_0 = a_0^2$，$A_n = a_n^2+b_n^2$，$B_n=0$.

易知 $f(x)$ 一致连续 ⇐ $f(x)$ 周期且连续

设 M 是 f 上界，对 $\forall x$ 都有 $|f(x)|\le M$.

对 $\forall\varepsilon>0$，$\exists\delta$，当 $|x_1-x_2|<\delta$ 时，有 $|f(x_1)-f(x_2)|<\frac{\varepsilon}{2M}$.

$\therefore$ 当 $|x_1-x_2|<\delta$ 时，

$|F(x_1)-F(x_2)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi}f(t)[f(x_1+t)-f(x_2+t)]\,dt\right|$

$\le\frac{1}{\pi}\int_{-\pi}^{\pi}|f(t)|\,|f(x_1+t)-f(x_2+t)|\,dt$

$\le\frac{1}{\pi}\cdot M\cdot\frac{\varepsilon}{2M}\cdot 2\pi = \varepsilon$

$\therefore F(x)$ 在 R 上连续，以 2π 为周期.

$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt\,dx$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt\,dx$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)\int_{-\pi+t}^{\pi+t}f(u)\,du\,dt$　周期长度的积分

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)\,dt\int_{-\pi}^{\pi}f(u)\,du$.　　都为 a_0.

$= \left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx\right)^2 = a_0^2$

$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt\cos nx\,dx$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\cos nx\,dx\,dt$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)\int_{-\pi}^{\pi}f(u)[\cos nu\cos nt + \sin nu\sin nt]\,du\,dt$

$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)\cos nt\int_{-\pi}^{\pi}f(u)\cos nu\,du\cdots = a_n^2+b_n^2.$

同理可得 $B_n=0$.

$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt\sin nx\,dx$

$\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\,dt = F(x) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}A_n\cos nx$

特别取 $x=0$，$\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(t)\,dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2).$

例. (综合T8) f 在 $[-\pi,\pi]$ 上连续，则该区间上可积且平方可积，若导数 f'，如果 $f(-\pi)=f(\pi)$，$\int_{-\pi}^{\pi}f(x)\,dx=0$.

证明：$\int_{-\pi}^{\pi}(f'(x))^2\,dx \ge \int_{-\pi}^{\pi}f^2(x)\,dx$ 成立.

$f(x)\sim\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$f'(x)\sim\frac{a_0'}{2} + \sum_{n=1}^{\infty}(a_n'\cos nx + b_n'\sin nx)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx = 0$，$a_0' = \frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\,dx = \frac{1}{\pi}f(x)\cdots = 0.$

$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos nx\,dx$

$= \frac{1}{\pi}\left[f(x)\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi}f(x)\sin nx\,dx\right] = nb_n$

$b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\sin nx\,dx$

$= \frac{1}{\pi}\left[f(x)\sin nx\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi}f(x)\cos nx\,dx\right] = -na_n$

$f(x)$，$f'(x)$ 连续且可积.

$\therefore\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx = \sum_{n=1}^{\infty}(a_n^2+b_n^2)$

$\frac{1}{\pi}\int_{-\pi}^{\pi}f'^2(x)\,dx = \sum_{n=1}^{\infty}(a_n'^2+b_n'^2) = \sum_{n=1}^{\infty}n^2(a_n^2+b_n^2)$

$\therefore\frac{1}{\pi}\int_{-\pi}^{\pi}f^2\,dx \le \frac{1}{\pi}\int_{-\pi}^{\pi}[f'(x)]^2\,dx$

取等：$n=1$　$n\ge2$ 时，$a_n=b_n=0$.

返回条件 ⇒ 若 $f(x)=\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.

则 $f(x) = a_1\cos x + b_1\sin x$.