第一型

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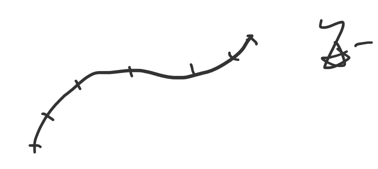
一、数量场在曲线上的积分

1. 定义



分一段，$\widehat{M_iM_{i-1}} = \Delta S_i$，

任取 $N_i(\xi_i, \eta_i, \gamma_i)$，

$\sum_i \varphi(N_i)\Delta S_i$

$\therefore \lim_{\lambda\to 0}\sum_{i=1}^{n}\varphi(\xi_i,\eta_i,\gamma_i)\Delta S_i$

2. 计算

正则曲线 $\vec{r}(t) = (x(t), y(t), z(t))$

$\varphi(x(t), y(t), z(t))$ 连续。

作分割，$ds = |\vec{r}'(t)|dt$，
$\Delta S_i = \int_{t_{i-1}}^{t_i}|\vec{r}'(t)|dt = |\vec{r}'(\theta_i)|\Delta t_i$（中值）　弧长。

$\sum_i \varphi(x(t_i), y(t_i), z(t_i))|\vec{r}'(\theta_i)|\Delta t_i$

不同于多格黎曼和 $\sum_i \varphi(x(t_i), y(t_i), z(t_i))|\vec{r}'(t_i)|\Delta t_i$

需证 $\sum_i \varphi(x,y,z)|\vec{r}'(\theta_i)|\Delta t_i \to \sum_i \varphi(x,y,z)|\vec{r}'(t_i)|\Delta t_i$

$|\vec{r}'(\theta_i) - \vec{r}'(t_i)| < \varepsilon$，一致连续故成立。

$\therefore$ 求和 → 黎曼和。

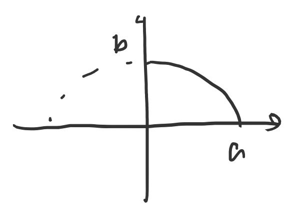
$S(t) = \int_\alpha^t |\vec{r}'(\tau)|d\tau$，$S'(t) = |\vec{r}'(t)| > 0$。

$$\int_L \varphi(x,y,z)ds = \int_\alpha^\beta \varphi(x(t), y(t), z(t))|\vec{r}'(t)|dt$$
$$= \int_\alpha^\beta \varphi(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

显式曲线 $y = y(x)$，$a \le x \le b$。
$$\int_L f(x,y)ds = \int_a^b f(x, y(x))\sqrt{1 + (y'(x))^2}\,dx.$$

极坐标 $r = r(\theta)$，$x = r(\theta)\cos\theta$，$y = r(\theta)\sin\theta$。
$$[x'(\theta)]^2 + [y'(\theta)]^2 = [r'(\theta)(\cos\theta - \sin\theta)]^2 + [r'(\theta)(\sin\theta + \cos\theta)]^2$$
$$= r^2(\theta) + r'(\theta)^2$$
$$\int_\alpha^\beta f(r\cos\theta, r\sin\theta)\sqrt{r^2 + r'^2}\,d\theta.$$

3. 例

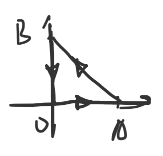


曲线积分 $\int_L xy\,ds$，L 为椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 在第一象限的弧长。

L 的参数方程 $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$

$$\int_L xy\,ds = \int_0^{\frac{\pi}{2}} a\cos\theta \cdot b\sin\theta\sqrt{(-a\sin\theta)^2 + (b\cos\theta)^2}\,d\theta$$
$$= \frac{ab}{2}\int_0^{\frac{\pi}{2}}\sqrt{b^2 + (a^2 - b^2)\sin^2\theta}\,d\sin^2\theta$$
$$= \frac{ab}{2}\int_0^1\sqrt{b^2 + (a^2 - b^2)t}\,dt$$
$$= \frac{2}{3}\times\frac{ab}{2}\times\frac{1}{(a^2 - b^2)}(b^2 + (a^2 - b^2)t)^{\frac{3}{2}}\Big|_0^1$$
$$= \frac{ab(a^2 + ab + b^2)}{3(a+b)}.$$

例. 求 $\int_L x\,ds$，其中 L



参数表示 $\overline{OA}$：$(x,y) = (t, 0)$，$t \in [0,1]$

$\overline{AB}$：$(x,y) = (t, 1-t)$

$\overline{BO}$：$(x,y) = (0, t)$

$$\int_L x\,ds = \int_0^1 t\,dt + \int_0^1 t\sqrt{2}\,dt + \int_0^1 0\cdot dt = \frac{\sqrt{2}+1}{2}.$$

二、数量场在曲面上的积分

1. 曲面的面积

① $\sigma(S_{ij}) \approx |\vec{r}_u'(u_i, v_j) \times \vec{r}_v'(u_i, v_j)|\Delta u_i \Delta v_j$.

$dS = |\vec{r}_u'(u,v) \times \vec{r}_v'(u,v)|\,du\,dv$

$$\sigma(S_{ij}) = \iint_S 1\,dS = \iint_D |\vec{r}_u'(u,v) \times \vec{r}_v'(u,v)|\,du\,dv$$
$$= \iint_D \sqrt{EG - F^2}\,du\,dv.$$

② 特别当曲面在平面上时。

若 xOy 平面。$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = 0 \end{cases}$

$\vec{r}_u' \times \vec{r}_v' = \left(0, 0, \frac{\partial(x,y)}{\partial(u,v)}\right)$，$|\vec{r}_u' \times \vec{r}_v'| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$（换元）

③ 显式曲面 $z = f(x,y)$，$\vec{r} = (x, y, f(x,y))$

$\vec{r}_x' = (1, 0, \frac{\partial f}{\partial x})$，$\vec{r}_y' = (0, 1, \frac{\partial f}{\partial y})$

$$\sigma S = \iint_D \sqrt{1 + (\tfrac{\partial f}{\partial x})^2 + (\tfrac{\partial f}{\partial y})^2}\,dx\,dy.$$

例. 求半径为 R 的球的表面积。

$x^2 + y^2 + z^2 = R^2$，

$x = R\sin\theta\cos\varphi$，$y = R\sin\theta\sin\varphi$，$z = R\cos\theta$。

$|\vec{r}_\theta| = \sqrt{(R\cos\varphi\cos\theta)^2 + (R\sin\varphi\cos\theta)^2 + (R\sin\theta)^2} = R$.　$\vec{r}_\theta = (\quad)$

$|\vec{r}_\varphi| = \sqrt{(-R\sin\theta\sin\varphi)^2 + (R\sin\theta\cos\varphi)^2 + 0^2} = R\sin\theta$　$\vec{r}_\varphi = (\quad)$

$F = 0$，$E = R^2\sin^2\theta$，$G = R^2$.

$$S = \int_0^{2\pi}\int_0^{\pi}\sqrt{R^2R^2\sin^2\theta}\,d\theta\,d\varphi = R^2 \times 2\pi\int_0^{\pi}\sin\theta\,d\theta = 4\pi R^2.$$

例. 求球面 $x^2 + y^2 + z^2 = R^2$ 被柱面 $x^2 + y^2 = Rx$ 所截下的曲面面积。

y、z 正负对称 ⇒ $y \ge 0$，$z \ge 0$，最后乘4.　$y \ge 0$，$z \ge 0$，$\varphi, \theta \in [0, \frac{\pi}{2}]$.

$x = R\sin\theta\cos\varphi$，$y = R\sin\theta\sin\varphi$，$z = R\cos\theta$.

$R^2\sin^2\theta \le R^2\sin\theta\cos\varphi$，$\sin\theta \le \cos\varphi = \sin(\frac{\pi}{2} - \varphi)$ ⇒ $0 \le \theta \le \frac{\pi}{2} - \varphi$.

$$S = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}-\varphi} R^2\sin\theta\,d\theta\,d\varphi \times 4 \quad \left(\text{或}\int_0^{\frac{\pi}{2}}\sin\theta\int_0^{\frac{\pi}{2}-\theta}d\varphi\,d\theta\right)$$
$$= 4R^2\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}-\varphi}\sin\theta\,d\theta$$
$$= 4R^2\int_0^{\frac{\pi}{2}}\cos\theta\Big|_{\frac{\pi}{2}-\varphi}^{0}d\varphi = 4R^2(\frac{\pi}{2} - 1).$$

例. 计算 $y^2 + z^2 = x^2$ 被柱面 $y^2 + z^2 = R^2$ 截的面积。

$x = \sqrt{y^2 + z^2}$ 显式，两半。

$$S = 2 \times \iint\sqrt{1 + (\tfrac{\partial x}{\partial y})^2 + (\tfrac{\partial x}{\partial z})^2}\,dy\,dz$$
$$= 2\sqrt{2} \times \pi R^2.$$

2. 计算

$M_i(\xi_i, \eta_i, \gamma_i) \in S_i$，

$$\lim_{\lambda\to 0}\sum_{i=1}^{n}\varphi(\xi_i, \eta_i, \gamma_i)\sigma(S_i) = \iint_S \varphi(x,y,z)\,dS$$
$$= \iint_S \varphi(x(u,v), y, z)\sqrt{EG - F^2}\,du\,dv.$$

例. 设 S 为第一卦限的球面 $x^2 + y^2 + z^2 = R^2$，计算曲面积分 $\iint_S x^2 + y^2\,dS$.

S_1. $\theta \in [0, \frac{\pi}{2}]$，$\varphi \in [0, \frac{\pi}{2}]$.

则 $\iint_S (x^2 + y^2)\,dS = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} R^2\sin^2\theta \times R^2\sin\theta\,d\varphi\,d\theta$
$$= \frac{\pi}{2}R^4 \times \int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta = \frac{\pi}{3}R^4.$$

S_2. 对称，$\iint_S x^2\,dS = \iint_S y^2\,dS = \iint_S z^2\,dS$

$\therefore$ 原积分 $= \frac{2}{3}\iint_S x^2 + y^2 + z^2\,dS = \frac{2}{3}R^2 \times \frac{1}{8} \times 4\pi R^2$

例. 设 S 是 $z^2 = k^2(x^2 + y^2)$ 被柱面 $x^2 + y^2 = 2ax$ 所截得曲面，计算 $\iint_S (y^2x^2 + z^2x^2 + x^2y^2)\,dS$

解：显式 $z = k\sqrt{x^2 + y^2}$.

$$\iint_S (k^2(x^2 + y^2)^2 + x^2y^2)\sqrt{1 + k^2\frac{x^2}{x^2+y^2} + k^2\frac{y^2}{x^2+y^2}}\,dx\,dy$$
$$= \iint_S (k^2(x^2 + y^2)^2 + x^2y^2)\sqrt{1 + k^2}\,dx\,dy.$$

极坐标换元. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\varphi}(k^2r^4 + r^4\sin\varphi\cos\varphi)\sqrt{1+k^2}\,r\,dr\,d\varphi$

$$= \frac{\sqrt{1+k^2}}{6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\varphi} r^5(k^2 + \sin\varphi\cos\varphi)\,dr\,d\varphi$$
$$= \frac{\sqrt{1+k^2}}{6}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2a\cos\varphi)^6(k^2 + \sin\varphi\cos\varphi)\,d\varphi$$
$$= \frac{64a^6\sqrt{1+k^2}}{3}\int_0^{\frac{\pi}{2}}(k^2\cos^6\varphi + \cos^8\varphi - \cos^{10}\varphi)\,d\varphi$$

例. 若曲面 S 的球坐标表示
$\begin{cases} x = r(\theta)\sin\theta\cos\varphi \\ y = r(\theta)\sin\theta\sin\varphi \\ z = r(\theta)\cos\theta \end{cases}$　$(\theta, \varphi) \in D$，$r \in C^1$.

求证：曲面 S 面积为 $\sigma(S) = \iint_D \sqrt{r^2 + (r')^2}\,r\sin\theta\,d\theta\,d\varphi$.

$\vec{r}_\theta' = (r'(\theta)\sin\theta\cos\varphi + r(\theta)\cos\theta\cos\varphi,\ r'(\theta)\sin\theta\sin\varphi + r(\theta)\cos\theta\sin\varphi,\ r'(\theta)\cos\theta - r(\theta)\sin\theta)$

$\vec{r}_\varphi' = (-r(\theta)\sin\theta\sin\varphi,\ r(\theta)\sin\theta\cos\varphi,\ 0)$.

$E = (r'(\theta)\sin\theta)^2 + (r(\theta)\cos\theta)^2 + 2r'(\theta)r(\theta)\sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi)$
$\quad - 2r'(\theta)r(\theta)\cos\theta\sin\theta + (r'(\theta)\cos\theta)^2 + (r(\theta)\sin\theta)^2$
$\quad = r'^2(\theta) + r^2(\theta)$

$G = r^2(\theta)\sin^2\theta\sin^2\varphi + r^2(\theta)\sin^2\theta\cos^2\varphi = r^2(\theta)\sin^2\theta$.

$F = -r'(\theta)r(\theta)\sin^2\theta\cos\varphi\sin\varphi - r^2(\theta)\sin\theta\cos\theta\sin\varphi\cos\varphi$
$\quad + r'(\theta)r(\theta)\sin^2\theta\cos\varphi\sin\varphi + r^2(\theta)\sin\theta\cos\theta\sin\varphi\cos\varphi = 0$.　正交。

$$\sigma S = \iint_D\sqrt{EG - F^2}\,d\theta\,d\varphi = \iint_D\sqrt{r^2 + (r')^2}\,r\sin\theta\,d\theta\,d\varphi.$$

例. 计算积分 $A = \iiint_V z\,dV$

其中 V 是球面 $x^2 + y^2 + z^2 = 2az$ 和 $x^2 + y^2 + z^2 = az$ 之间的立体。

$A = \iiint_{V_1} z\,dV - \iiint_{V_2} z\,dV$.　第一个球 $(0, 0, a)$，第二个球 $(0, 0, \frac{a}{2})$.

$$= \iiint_{V_1}(z - a)\,dV + \iiint_{V_1}a\,dV - \iiint_{V_2}(z - \tfrac{a}{2})\,dV - \iiint_{V_2}\tfrac{a}{2}\,dV$$

（$\iiint_{V_1}(z-a)\,dV$：对称，为零。　$\iiint_{V_2}(z-\frac{a}{2})\,dV$：对称，为零。）