曲线积分和曲面积分

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$F(x)=\int f(x)dx + C$, $dF = f(x)dx$, $F(x)=\int_{x_0}^{x} f(t)dt$

$F(x,y,z)$ 满足 $dF = F_x dx + F_y dy + F_z dz$

$F(x,y)=\int_{(x_0,y_0)}^{(x,y)} F_x dx + F_y dy$

一、第一型曲线积分

（数量场在曲线上的积分）

1. 设 R^3 中有一条曲线 L: $\vec r(t)=x(t)\vec i+y(t)\vec j+z(t)\vec k$, $\varphi(x,y,z)$ 定义在包含 L 的区域内连续.

记 Δs_i 是 $M_{i-1}M_i$ 的弧长, $M_i(x(t_i), y(t_i), z(t_i))$.

ξ_i, η_i, ζ_i 是弧上的任意一点.

考虑 $\sum \varphi(\xi_i,\eta_i,\zeta_i)\Delta s_i = \int_L \varphi(x,y,z)\, ds$

$s = \int_\alpha^t |\vec r'(t)| dt$.

$\Delta s_i = \int_{t_{i-1}}^{t_i} |\vec r'(t)| dt = |\vec r'(\theta_i)| \Delta t_i$

$I = \sum \varphi(\xi_i,\eta_i,\zeta_i)|\vec r'(\theta_i)|\Delta t_i$

$= \sum \varphi(x(\tau_i), y(\tau_i), z(\tau_i)) |\vec r'(\theta_i)| \Delta t_i$ $(\tau_i \in (t_{i-1},t_i))$

$= \sum \varphi(x(\tau_i),y(\tau_i),z(\tau_i)) |\vec r'(\tau_i)| \Delta t_i + \sum \varphi(x(\tau_i),y(\tau_i),z(\tau_i))\underline{(|\vec r'(\theta_i)|-|\vec r'(\tau_i)|)}\Delta t_i$

记 $I_1 + I_2$.

由于 $\varphi(x,y,z)$ 连续, 可积,

$I_1 \to \int_\alpha^\beta \varphi(x(t),y(t),z(t))|\vec r'(t)| dt$

(较难证明 $I_2 \to 0$.) φ 有界 $\Rightarrow |\varphi|<M$,

$|\vec r'(t)|$ 在 $[\alpha,\beta]$ 上连续, 一致连续.

$\forall \varepsilon>0, \exists \delta$, $t_1,t_2\in[\alpha,\beta]$, $|t_1-t_2|<\delta$,

都有 $||\vec r'(t_1)|-|\vec r'(t_2)||<\varepsilon$

当 $|T|<\delta$ 时, $|\tau_i-\theta_i| \le |T| < \delta$.

$||\vec r'(\tau_i)|-|\vec r'(\theta_i)||<\varepsilon$

$\therefore |I_2| \le \sum|\varphi|\,|\vec r'(\tau_i)-\vec r'(\theta_i)|\Delta t_i < M\varepsilon \sum \Delta t_i = M\varepsilon(\beta-\alpha)$, $I_2\to 0$

2. 结论: 设 L: $\vec r=\vec r(t)$ 光滑, $\varphi(x,y,z)$ 连续,

则 $\int_L \varphi\, ds = \int_\alpha^\beta \varphi(x(t),y(t),z(t))|\vec r'(t)|\, dt$

特别 L: $y=f(x)$, $x\in[a,b]$.

$\vec r(x)=x\vec i+f(x)\vec j+0\vec k$, $\vec r'(x)=\vec i+f'(x)\vec j+0$.

$\therefore \int_L \varphi(x,y)\, ds = \int_a^b \varphi(x,f(x)) \sqrt{1+f'^2}\, dx$

L: $r=r(\theta)$

$\vec r(\theta) = x(\theta)\vec i + y(\theta)\vec j = r(\theta)\cos\theta\, \vec i + r(\theta)\sin\theta\, \vec j$

$\vec r'(\theta) = (r'(\theta)\cos\theta - r(\theta)\sin\theta)\vec i + (r'(\theta)\sin\theta + r(\theta)\cos\theta)\vec j$

$\therefore \int_L \varphi(x,y)\, ds = \int_\alpha^\beta \varphi(r(\theta)\cos\theta, r(\theta)\sin\theta) \sqrt{r^2+r'^2}\, d\theta$.

3. 例题

例. 设 L: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(x\ge 0, y\ge 0)$. 求 $\int_L xy\, ds$.

$x=a\cos\theta$, $y=b\sin\theta$, $0\le\theta\le\frac{\pi}{2}$

$\vec r = a\cos\theta\, \vec i + b\sin\theta\, \vec j$, $\vec r' = -a\sin\theta\, \vec i + b\cos\theta\, \vec j$.

$|\vec r'(\theta)| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$\int_L xy\, ds = \int_0^{\frac{\pi}{2}} ab\cos\theta\sin\theta \sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\, d\theta$

$= \frac{ab}{2}\int_0^{\frac{\pi}{2}} \sqrt{b^2+(a^2-b^2)\sin^2\theta}\, d\sin^2\theta = \dots$

例. $\int_L x\, ds = \int_{L_1} x\, ds + \int_{L_2} x\, ds + \int_{L_3} x\, ds$. L:

L_1: $x(t)=t$, $y(t)=0$.

$\vec r=t\vec i+0\vec j$, $\vec r'=\vec i$

$\int_{L_1} x\, ds = \int_0^1 t\, dt = \frac{1}{2}$

L_2: $x(t)=1-t$, $y(t)=t$

$\vec r = (1-t)\vec i + t\vec j$, $\vec r' = -\vec i+\vec j$

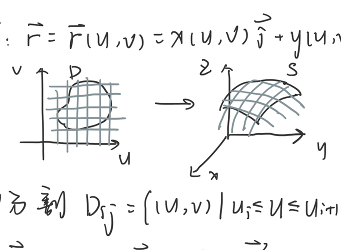
$\int_{L_2} x\, ds = \int_0^1 (1-t)\sqrt{2}\, dt =$

$\int_{L_3} x\, ds = 0$.

二、第一型曲面积分

（数量场在曲面上的积分）

1. 设 S: $\vec r = \vec r(u,v) = x(u,v)\vec i + y(u,v)\vec j + z(u,v)\vec k$



D 的分割 $D_{ij} = \{(u,v) \mid u_i \le u \le u_i+\Delta u_i, v_j \le v \le v_j+\Delta v_j\}$

向量 $\vec r(u_{i+1},v_j) - \vec r(u_i,v_j) = \vec r'_u(u_i,v_j)\Delta u_i + o(\Delta u_i)$

$\vec r(u_i,v_{j+1}) - \vec r(u_i,v_j) = \vec r'_v(u_i,v_j)\Delta v_j + o(\Delta v_j)$

$\Delta S_{ij} \approx |\vec r'_u(u_i,v_j)\Delta u_i \times \vec r'_v(u_i,v_j)\Delta v_j|$

$= |\vec r'_u \times \vec r'_v|\, \Delta u_i \Delta v_j$

$\sum \varphi(\xi_{ij},\eta_{ij},\zeta_{ij})\Delta S_{ij} = \sum \varphi(x(\xi_{ij},\eta_{ij}), y(\xi_{ij},\eta_{ij}), z(\xi_{ij},\eta_{ij}))|\vec r'_u\times \vec r'_v|\Delta u_i\Delta v_j$

$\Rightarrow \iint_S \varphi(x,y,z)\, dS = \iint_D \varphi(x(u,v),y(u,v),z(u,v))|\vec r'_u\times\vec r'_v|\, du\, dv$

其中 $|\vec r'_u\times\vec r'_v|$, $|\vec r'_u\times\vec r'_v|^2 = \underbrace{(\vec r'_u\cdot\vec r'_u)}_{E}\underbrace{(\vec r'_v\cdot\vec r'_v)}_{G} - \underbrace{(\vec r'_u\cdot\vec r'_v)^2}_{F}$

若 S 在平面上, 即 $x=x(u,v)$, $y=y(u,v)$, $z=0$.

通过 $dS = \frac{\partial(x,y)}{\partial(u,v)} dx\,dy$

$\iint_S \varphi\, dS = \iint_D \varphi(x,y,z) \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$ (变量代换).

2. 结论

设 $\varphi(x,y,z)$ 在 $V\subset R^3$ 上连续,

$S\subset D$ 且为光滑曲面, $(\vec r'_u\times\vec r'_v \ne 0)$

S: $\vec r = \vec r(u,v) = x(u,v)\vec i + y(u,v)\vec j + z(u,v)\vec k$

则 $\iint_S \varphi\, dS = \iint_D \varphi(x(u,v),y(u,v),z(u,v))|\vec r'_u\times\vec r'_v|\, du\, dv$.

3. 例. 计算半径为 R 的球的表面积.

S: $\vec r(\theta,\varphi) = R(\sin\varphi\cos\theta\, \vec i + \sin\varphi\sin\theta\, \vec j + \cos\varphi\, \vec k)$

$\vec r'_\theta = R(-\sin\varphi\sin\theta\, \vec i + \sin\varphi\cos\theta\, \vec j + 0\vec k)$

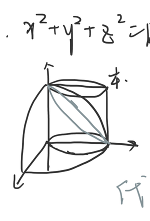
$\vec r'_\varphi = R(\cos\varphi\cos\theta\, \vec i + \cos\varphi\sin\theta\, \vec j - \sin\varphi\, \vec k)$

$E = \vec r'_\theta\cdot\vec r'_\theta = R^2\sin^2\varphi$, $G = \vec r'_\varphi\cdot\vec r'_\varphi = R^2$, $F=0$.

$|\vec r'_\theta\times\vec r'_\varphi| = \sqrt{EG-F^2} = R^2\sin\varphi$.

$\therefore \sigma(S) = \iint_S dS = \iint_{\substack{0\le\theta\le 2\pi \\ 0\le\varphi\le\pi}} R^2\sin\varphi\, d\theta\, d\varphi = \int_0^{2\pi} d\theta \int_0^\pi R^2\sin\varphi\, d\varphi = 4\pi R^2$.

例. $x^2+y^2+z^2=R^2$ 被 $x^2+y^2=Rx$ 截下的面积.



S: $\vec r = R(\sin\varphi\cos\theta\, \vec i + \sin\varphi\sin\theta\, \vec j + \cos\varphi\, \vec k)$

$0\le\theta\le\frac{\pi}{2}$, $0\le\varphi\le\frac{\pi}{2}$

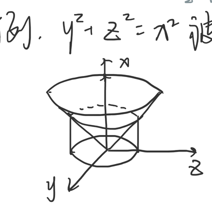
代入柱面, $R^2\sin^2\varphi = R^2\sin\varphi\cos\theta$

$\therefore \sin\varphi = \cos\theta$, $\theta = \frac{\pi}{2}-\varphi$

$\sigma(S) = 4\iint R^2\sin\varphi\, d\theta\, d\varphi = 4R^2\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}-\varphi} \sin\varphi\, d\theta$

$= 2R^2(\pi-2)$

例. $y^2+z^2=x^2$ 被 $y^2+z^2=2x$ 截下的面积.



$x=\sqrt{y^2+z^2}$, $y^2+z^2\le R^2$.

$\vec r(y,z) = \sqrt{y^2+z^2}\, \vec i + y\vec j + z\vec k$

$\vec r'_y = \frac{y}{\sqrt{y^2+z^2}}\vec i + \vec j$, $\vec r'_z = \frac{z}{\sqrt{y^2+z^2}}\vec i + \vec k$

$\sqrt{EG-F^2} = \sqrt{1+x_y'^2+x_z'^2}\, dy\, dz$. $S = 2\sqrt{2}\pi$

例. S: $x^2+y^2+z^2=R^2$ 第一象限.

计算 $\varphi(x,y,z) = x^2+y^2$ 在 S 上的积分.

$\iint_S (x^2+y^2)\, dS$. S: $\vec r = R(\sin\varphi\cos\theta\, \vec i + \sin\varphi\sin\theta\, \vec j + \cos\varphi\, \vec k)$

$= \iint R^2\sin^2\varphi \times R^2\sin\varphi\, d\theta\, d\varphi = R^4\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin^3\varphi\, d\varphi$

$= \frac{\pi}{2}R^4\int_0^{\frac{\pi}{2}}(1-\cos^2\varphi)\sin\varphi\, d\varphi = \dots$

例. 设 S: $z=f(x,y)$, $(x,y)\in D$. 求 $\iint_S \varphi(x,y,z)\, dS$

S: $\vec r = x\vec i + y\vec j + f(x,y)\vec k$

$\vec r'_x = \vec i + f'_x\vec k$, $\vec r'_y = \vec j + f'_y\vec k$

$\sqrt{EG-F^2} = \sqrt{1+f_x'^2+f_y'^2}$

$\iint_S \varphi\, dS = \iint_D \varphi(x,y,f(x,y))\sqrt{1+f_x'^2+f_y'^2}\, dx\, dy$

三、第二型曲线积分

（向量场在曲线上的积分）

1. 回顾.

$\int_L f\, ds = \int_\alpha^\beta f(x(t),y(t),z(t))|\vec r'(t)|\, dt$

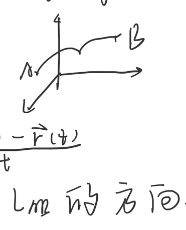
$\iint_S f\, dS = \iint_D f(x(u,v),y(u,v),z(u,v))|\vec r'_u\times\vec r'_v|\, du\, dv$

2. 曲线定向.

① L_{AB}: $\vec r = \vec r(t) = x(t)\vec i + y(t)\vec j + z(t)\vec k$

$\alpha\le t\le\beta$,

t 为正向参数.



$\vec r'(t) = \lim_{\Delta t\to 0} \frac{\vec r(t+\Delta t)-\vec r(t)}{\Delta t}$

$\vec r'(t)$ 的方向是 L_{AB} 的方向

② 逐段光滑（同上）

③ 封闭曲线, 逆时针为正方向.

3. 计算.

$\sum \vec F_i\cdot\Delta\vec r_i = \sum(P_i\Delta x_i + Q_i\Delta y_i + R_i\Delta z_i)$

$= \sum P(\xi_i,\eta_i,\zeta_i)\Delta x_i + Q_i\Delta y_i + R_i\Delta z_i$

极限存在, $\int_L P(x,y,z)dx + Q\,dy + R\,dz$

($\vec v = P\vec i + Q\vec j + R\vec k$ 的一次微分形式).

设 L_{AB}: $\vec r = x(t)\vec i + y(t)\vec j + z(t)\vec k$

$\Delta\vec r_i = \Delta x_i\vec i + \Delta y_i\vec j + \Delta z_i\vec k$

$= (x(t_i)-x(t_{i-1}))\vec i + (y\cdots)\vec j + (z\cdots)\vec k$

$= (x'(\lambda_i)\vec i + y'(\mu_i)\vec j + z'(\nu_i)\vec k)\Delta t_i$

$\sum \vec F_i\cdot\Delta\vec r_i = \sum(P(\xi_i,\eta_i,\zeta_i)x'(\lambda_i) + Qy' + Rz')\Delta t_i$

$\to \int_\alpha^\beta (P(x(t),y(t),z(t))x'(t) + Qy' + Rz')\, dt$

$= \int_\alpha^\beta \vec F\cdot\vec r'(t)\, dt = \int_\alpha^\beta \vec F\cdot d\vec r$.

结论: 设 $\vec v = P(x,y,z)\vec i + Q(x,y,z)\vec j + R(x,y,z)\vec k$

连续, L_{AB}: $\vec r = \vec r(t) = x(t)\vec i+y(t)\vec j+z(t)\vec k$,

则 $\int Pdx+Qdy+Rdz = \int_\alpha^\beta (\vec v\cdot\vec r'(t))\, dt = \int_\alpha^\beta \vec v\, d\vec r$.

记 $|\vec r'(t)|dt = ds$, $\frac{\vec r'(t)}{|\vec r'(t)|} = \vec\tau(t)$

$\int_{L_{AB}} \vec v\cdot d\vec r = \int_{L_{AB}} \vec v\cdot\vec r'(t)\, dt = \int_{L_{AB}} \vec v\cdot\vec\tau\, ds$

数量场 $\vec v\cdot\vec\tau$ 在 L_{AB} 上的曲线积分

若 $A=B$, L_{AB} 是简单闭曲线,

则 $\int_{L_{AB}} \vec v\cdot d\vec r = \oint_L \vec v\cdot d\vec r$, $\vec v$ 在 L 上的环量.

4. 性质.

① $\int(\alpha\vec v_1+\beta\vec v_2)\, d\vec r = \alpha\int\vec v_1\, d\vec r + \beta\int\vec v_2\, d\vec r$

② 积分区间可加.

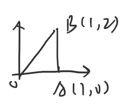
③ $\int_{L_{AB}} \vec v\, d\vec r = -\int_{L_{BA}} \vec v\, d\vec r$.

④ 记 $\frac{\vec r'(t)}{|\vec r'(t)|} = \vec\tau(t) = \cos\alpha(t)\vec i + \cos\beta(t)\vec j + \cos\gamma(t)\vec k$

$\int_{L_{AB}} \vec v\cdot d\vec r = \int_{L_{AB}} \vec v\cdot\vec\tau\, ds = \int_{L_{AB}} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\, dt$

$= \int_{L_{AB}} Pdx + Qdy + Rdz$

5. 例. 计算 $\oint_L xy\,dx + x^2\,dy$,



$\vec v = xy\vec i + x^2\vec j$, $P(x,y)=xy$, $Q(x,y)=x^2$.

$\oint_L \vec v = \int_{OA} + \int_{AB} + \int_{BO}$

L_{OA}: $x=x(t)=t$, $y(t)=0$, $0\le t\le 1$

$\int_{OA} (x(t)y(t)x'(t) + x^2(t)y'(t))\, dt = 0$.

L_{AB}: $x=x(t)=1$, $y=y(t)=t$, $0\le t\le 2$

$\int_{AB} (x(t)y(t)x'(t) + x^2(t)y'(t))\, dt$

$= \int_0^2 1\times 1\, dt = 2$.

L_{BO}: $y=2x \Rightarrow x=t$, $y=2t$, $0\le t\le 1$.

$\int_{BO} \vec v\, d\vec r = -\int_{OB} \vec v\cdot d\vec r$

$= -\int_0^1 (x(t)y(t)x'(t) + x^2(t)y'(t))\, dt$

$= -\int_0^1 (t\cdot 2t\times 1 + t^2\times 2)\, dt = -\frac{4}{3}$.

$\therefore \oint_L = 2-\frac{4}{3} = \frac{2}{3}$.

例. $\vec F = y\vec i - x\vec j + z\vec k$,

L_{AB}: $x=R\cos t$, $y=R\sin t$, $z=kt$. $0\le t\le 2\pi$.

$A(R,0,0)$, $B(R,0,2\pi k)$

$\int_{L_{AB}} \vec F\, d\vec r$

$= \int_0^{2\pi}(y(t)x'(t) - x(t)y'(t) + z(t)z'(t))\, dt$

$= 2\pi(\pi k^2 - R^2)$.

例. $\vec F = -k\frac{mM}{r^3}\vec r$, $r=|\vec r|$. $\int_{L_{AB}} \vec F\, d\vec r$.

$\int_{L_{AB}} \vec F\cdot d\vec r = -kmM\int_{L_{AB}} \frac{\vec r}{r^3}\cdot\vec r'\, dt$

$= -kmM\int \frac{rr'}{r^3}\, dt$ $\quad (\vec r\cdot\vec r)' = 2\vec r\cdot\vec r'$, $(r^2)' = 2rr'$

$= -kmM\int \frac{r'}{r^2}\, dt$

$= kmM\left.\frac{1}{r}\right|_{r_A}^{r_B} = kmM\left(\frac{1}{r_B}-\frac{1}{r_A}\right)$

例. $\vec v_1 = \frac{-y\vec i + x\vec j}{x^2+y^2}$. $\vec v_2 = \frac{x\vec i + y\vec j}{x^2+y^2}$.

L: $x^2+y^2=R^2$

$x=R\cos t$, $y=R\sin t$, $0\le t\le 2\pi$.

$\oint_L \vec v_1\, d\vec r = 2\pi$, $\oint_L \vec v_2\, d\vec r = 0$.

6. Green 定理.

① $\int_{L_{AB}} \vec v\, d\vec r = \int_{L_{AB}} \vec v\cdot d\vec r = -\int_{L_{BA}} \vec v\cdot d\vec r \Leftrightarrow \oint_{\square} \vec v\, d\vec r = 0$.

$\oint_{\square} \vec v\, d\vec r = \oint_{\square} P\,dx + \oint_{\square} Q\,dy$

$\oint_{\square} P\,dx = \int_a^b P(x,c)\, dx - \int_a^b P(x,d)\, dx = \int_a^b P(x,c)-P(x,d)\, dx$

$= -\int_a^b\int_c^d \frac{\partial P(x,y)}{\partial y}\, dy\, dx = -\iint_D \frac{\partial P}{\partial y}\, dx\, dy$.

$\oint_{\square} Q\,dy = \int_c^d Q(b,y) - Q(a,y)\, dy = \iint_D \frac{\partial Q}{\partial x}\, dx\, dy$

$\therefore \oint_{\partial D} \vec v\, d\vec r = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\, dy$.

② 设 D 是逐段光滑封闭曲线围成的区域, $L=\partial D$. $\vec v = P(x,y)\vec i + Q(x,y)\vec j$, 则

$\int_{\partial D} \vec v\, d\vec r = \int_{\partial D} P\,dx + Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\, dy$.

证明: $D_x = \{(x,y) \mid a\le x\le b, \varphi_1(x)\le y\le\varphi_2(x)\}$.

$\int P\,dx = \iint \frac{\partial P}{\partial y}\, dx\, dy$

$D_y = \{(x,y) \mid c\le y\le d, \psi_1(y)\le x\le\psi_2(y)\}$.

$\int Q\,dy = \iint \frac{\partial Q}{\partial x}\, dy\, dx$

D 分成多个 D_x. D_y.

$\int_{\partial D} P\,dx + Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\, dy$.

边界: 正反抵消.

③ 推论: D 的面积 $= \begin{cases} \frac{1}{2}\int_{\partial D}(-y\,dx + x\,dy) \\ \int_{\partial D} x\,dy \,/\, -\int_{\partial D} y\,dx \end{cases}$

取向量场 $\left(\begin{array}{l} \vec v = (-y\vec i + x\vec j)\frac{1}{2} \\ \vec v = 0\vec i + x\vec j \,/\, -y\vec i + 0\vec j \end{array}\right)$

7. D 是单连通区域, $\vec v = P\vec i + Q\vec j$ 光滑,

则 $\oint_L \vec v\, d\vec r = 0$, L 是 D 内光滑封闭区域.

$\Leftrightarrow$ $\vec v$ 是一个梯度场, $\exists\varphi(x,y)$ 使 $\vec v = \nabla\varphi$.

$\Leftrightarrow$ P 和 Q 在 D 中满足 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

证明: ①⇒② 设 $\oint_L \vec v\cdot d\vec r = 0$,

则积分与路径无关.

定义 $(x,y) \to \int_{L_{AB}} \vec v\cdot d\vec r = \int_{(x_0,y_0)}^{(x,y)} \vec v\, d\vec r$

$= \int_A^B \vec v\, d\vec r \triangleq \varphi(x,y)$

$\frac{\varphi(x+\Delta x, y) - \varphi(x,y)}{\Delta x} = \int_{(x,y)}^{(x+\Delta x,y)} \vec v\, d\vec r \times \frac{1}{\Delta x}$

$= \frac{1}{\Delta x}\int_{(x,y)}^{(x+\Delta x,y)} P\,dx + Q\,dy = \frac{1}{\Delta x}\int_x^{x+\Delta x} P(x,y)\, dx$

$\therefore \frac{\partial\varphi}{\partial x} = P$, $\frac{\partial\varphi}{\partial y} = Q$.

②⇒③ 设 $\vec v = \nabla\varphi$, $P = \frac{\partial\varphi}{\partial x}$, $Q = \frac{\partial\varphi}{\partial y}$.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Leftrightarrow \frac{\partial\varphi}{\partial x\partial y} - \frac{\partial\varphi}{\partial x\partial y} = 0$

③⇒① Green 公式.

8. 若 $\exists\varphi(x,y)$ 使 $\vec v = \nabla\varphi$, 则 $\int_{L_{AB}} \vec v\, d\vec r = \varphi(B) - \varphi(A)$.

设 $\vec v = P\vec i + Q\vec j$ 在 D 上积分与路径无关,

$\varphi(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec v\, d\vec r$

$\int_{L_{AB}} \vec v\, d\vec r = \int_A^{(x_0,y_0)} + \int_{(x_0,y_0)}^B = \int_{(x_0,y_0)}^B - \int_{(x_0,y_0)}^A = \varphi(B) - \varphi(A)$.

四、第二型曲面积分

（向量场在曲面上的积分）.

(1) 有向曲面

1. 单/双侧曲面.

光滑曲面, 各点取法向量.

法向量方向在该点不变 ⇒ 双侧的.

一点双侧, 点点双侧.

2. 曲面的定向

设 S: $\vec r = \vec r(u,v) = x(u,v)\vec i + y(u,v)\vec j + z(u,v)\vec k$.

选定 $\vec n = \vec r'_u\times\vec r'_v$ 为正向.

特别当 S: $z=f(x,y)$ $(x,y)\in D$.

$\vec r(x,y) = x\vec i + y\vec j + f(x,y)\vec k$ (显式表示)

$\vec r'_x = \vec i + 0 + f'_x\vec k$, $\vec r'_y = 0 + \vec j + f'_y\vec k$.

$\vec n = \frac{-f'_x\vec i - f'_y\vec j + \vec k}{\sqrt{1+f_x'^2+f_y'^2}}$, $\vec n\cdot\vec k$ 正负 → 方向

封闭曲面 $\vec n = \frac{\vec r}{R}$, 朝外为正.

3. 定向的协调性

右手定则 ⇒ 曲面边界的方向

4. 有向面积 $\Delta\vec S = \vec n\Delta S$.

S: $\vec r = \vec r(u,v)$, $dS = |\vec r'_u\times\vec r'_v|\, du\, dv$.

$d\vec S = \vec n\, dS = \frac{\vec r'_u\times\vec r'_v}{|\vec r'_u\times\vec r'_v|}|\vec r'_u\times\vec r'_v|\, du\, dv$

$= \vec r'_u\times\vec r'_v\, du\, dv$.

(2) 曲面积分

1. 定义: 通量 $\iint_S \vec v\cdot\vec n\, dS = \iint_S \vec v\cdot d\vec S$.

例, $S = \{(x,0,z) \mid 1\le x\le 3, 0\le z\le 2\}$.

$\vec n = \vec j$, 求: $\vec v = \frac{-y\vec i + x\vec j}{x^2+y^2}$.

$\iint_S \vec v\cdot\vec n\, dS = \iint_S \frac{x}{x^2+y^2}\, dS = \iint_S \frac{1}{x}\, dS$

$= \int_1^3 dx\int_0^2 \frac{1}{x}\, dz$.

2. 性质.

① $\iint_S (\alpha\vec v_1 + \beta\vec v_2)\, d\vec S = \alpha\iint_S \vec v_1\, d\vec S + \beta\iint_S \vec v_2\, d\vec S$

② $\iint_{S_1+S_2} \vec v\, d\vec S = \iint_{S_1} \vec v\, d\vec S + \iint_{S_2} \vec v\, d\vec S$.

③ $\iint_{S_+} \vec v\, d\vec S = -\iint_{S_-} \vec v\, d\vec S$.

3. 计算

设 $\vec v = P\vec i + Q\vec j + R\vec k$ 是光滑向量场.

S: $\vec r = x(u,v)\vec i + y(u,v)\vec j + z(u,v)\vec k$. 参数曲面

$dS = |\vec r'_u\times\vec r'_v|\, du\, dv$

$\iint_S \vec v\, d\vec S = \iint_S \vec v\cdot d\vec S = \iint_S \vec v\cdot\vec n\, dS$

$= \iint_D \vec v\cdot\vec r'_u\times\vec r'_v\, du\, dv$.

$= \iint_D \begin{vmatrix} P & Q & R \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix} du\, dv$.

又 $d\vec S = \vec n\, dS = \left(\frac{\partial(y,z)}{\partial(u,v)}\vec i + \frac{\partial(z,x)}{\partial(u,v)}\vec j + \frac{\partial(x,y)}{\partial(u,v)}\vec k\right)du\, dv$

$= \underbrace{\frac{\partial(y,z)}{\partial(u,v)}du\,dv}_{dy\wedge dz}\vec i + \underbrace{\frac{\partial(z,x)}{\partial(u,v)}du\,dv}_{dz\wedge dx}\vec j + \underbrace{\frac{\partial(x,y)}{\partial(u,v)}du\,dv}_{dx\wedge dy}\vec k$

$d\vec S$ 在坐标面投影

$d\vec S = \cos\alpha\, dS\, \vec i + \cos\beta\, dS\, \vec j + \cos\gamma\, dS\, \vec k$.

$\iint_S \vec v\, d\vec S = \iint_S \vec v\cdot\vec n\, dS = \iint_S P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$

特殊形式 ① $z=f(x,y)$, $\vec r = x\vec i + y\vec j + f(x,y)\vec k$

$\iint_S \vec v\, d\vec S = \iint_D (-Pf'_x - Qf'_y + R)\, dx\, dy$.

② $\vec v = P\vec i + 0\vec j + 0\vec k$

$\iint_S \vec v\, d\vec S = \iint_S P\, dy\, dz$

$= \iint_D P(x,y,z)\frac{\partial(y,z)}{\partial(u,v)}\, du\, dv$

4. 例. $\vec v(x,y,z) = P(x,y,z)\vec i + Q(x,y,z)\vec j + R(x,y,z)\vec k$

通过 S: $x^2+y^2+z^2=a^2$ 外侧的通量.

球面 $\vec r = x\vec i + y\vec j + z\vec k$, 法向量 $\vec n = \frac{\vec r}{a}$.

例. $\vec v = \vec E = \frac{q}{r^3}\vec r$, $\vec r = x\vec i + y\vec j + z\vec k$, $r = |\vec r|$.

S: $x^2+y^2+z^2=R^2$, 方向: 外侧.

$\iint_S \vec v\, d\vec S = \iint_S \frac{q}{r^3}\vec r\cdot\frac{\vec r}{R}\, dS = \frac{q}{R}\iint_S \frac{1}{r}\, dS$

$= \frac{q}{R^2}\iint_S dS = \frac{q}{R^2}\times 4\pi R^2 = 4\pi q$.

例. 设 S: $x^2+y^2=a^2$, $-h\le z\le h$, $\vec v = P\vec i + Q\vec j + R\vec k$.

$\iint_S \vec v\, d\vec S = \iint_S (P\vec i + Q\vec j + R\vec k)\cdot\frac{x\vec i + y\vec j}{a}\, dS$

$= \iint_S (Px + yQ)\, dS \times \frac{1}{a}$.

令 $x = a\cos\theta$, $y = a\sin\theta$, $z = z$,

$= \frac{1}{a}\iint_S P\frac{\partial(y,z)}{\partial(\theta,z)} + Q\frac{\partial(z,x)}{\partial(\theta,z)} + R\frac{\partial(x,y)}{\partial(\theta,z)}$

$= \iint_{\substack{0\le\theta\le 2\pi \\ -h\le z\le h}} (a\cos\theta P + a\sin\theta Q)\, d\theta\, dz$

$= a\int_{-h}^{h}\int_0^{2\pi} \cos\theta P + \sin\theta Q\, d\theta\, dz$

若 $\vec v = \frac{q}{r^3}\vec r = \vec E$,

$\iint \vec E\cdot\vec n\, dS = q\int_{-h}^{h}\int_0^{2\pi} \frac{a^2\cos^2\theta}{\sqrt{a^2+z^2}^3} + \frac{a^2\sin^2\theta}{\sqrt{a^2+z^2}^3}\, d\theta\, dz$.

$= a^2 q\int_{-h}^{h}\int_0^{2\pi} \frac{d\theta\, dz}{\sqrt{a^2+z^2}^3} = 4\pi q\frac{h}{\sqrt{a^2+h^2}}$.

例. $\vec v = y(x-z)\vec i + x^2\vec j + (y^2+xz)\vec k$

S: $0\le x\le a$, $0\le y\le b$, $0\le z\le c$.

$\iint_S y(x-z)\, dy\, dz + x^2\, dz\, dx + (y^2+xz)\, dx\, dy$

有投影面积的部分.

$\iint_S y(x-z)\, dy\, dz = \iint_{S_前} + \iint_{S_后} = \iint_{\substack{0\le y\le b \\ 0\le z\le c}} y(a-z)\, dy\, dz + \iint_{\substack{0\le y\le b \\ 0\le z\le c}} y(0-z)\, dy\, dz$

例. $\iint_S x^3\, dy\, dz + y^3\, dz\, dx$, S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$z\ge 0$, 指向方向向上.

令 $x = a\sin\theta\cos\varphi$, $y = b\sin\theta\sin\varphi$, $z = c\cos\theta$.

$0\le\varphi\le 2\pi$, $0\le\theta\le\frac{\pi}{2}$.

$\frac{\partial(y,z)}{\partial(\theta,\varphi)} = bc\sin^2\theta\cos\varphi$, $\frac{\partial(z,x)}{\partial(\theta,\varphi)} = ac\sin^2\theta\sin\varphi$.

$\iint_S x^3\, dy\, dz = \iint a^3\sin^3\theta\cos^3\varphi\,(bc\sin^2\theta\cos\varphi)$

$= a^3bc\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \sin^5\theta\cos^4\varphi\, d\theta\, d\varphi = \frac{2}{5}\pi a^3 b$.

$\iint_S y^3\, dz\, dx = \frac{2}{5}\pi ab^3 c$.

$\vec v_1 = (P,0,0)$. $\vec v_2 = (0,Q,0)$. $\vec v_3 = (0,0,R)$

$\iint_S \vec v\cdot\vec n\, dS = \iint_S \vec v_1\cdot\vec n + \iint_S \vec v_2\cdot\vec n + \iint_S \vec v_3\cdot\vec n$.

$\iint_S \vec v_1\cdot\vec n\, dS = \iint_S P\, dy\, dz \Rightarrow$ 投影