课后题

2022年2月15日 星期二 上午11:31

Jacob 行列式.
面积：标准了就写出来。

习题 11.1

1.(1) $\int_0^{2\pi} \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\, dt = \int_0^{2\pi}\sqrt{3}e^t dt = \sqrt{3}(e^{2\pi}-1)$.

(2) $x=3t,\ y=3t^2,\ z=2t^3$. $\int_0^1 \sqrt{9+36t^2+36t^4}\,dt = \int_0^1 3\times(1+2t^2)\,dt = 3(t+\frac{2}{3}t^3)\Big|_0^1 = 5$.

(3) $\int_0^{\frac{\pi}{2}} \sqrt{a^2+a^2 r^2 t}\, dt \ldots = 2a\ln\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right| \to \frac{a}{2}\ln(3+2\sqrt{2})$.

(4) $z = r\omega\theta$. $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$ …

(5) $4ax = (y+z)^2$, $\frac{4}{3}x^2+y^2=z^2$ …

2.(1) $\int_0^{2\pi} a^2(1-\cos t)^2\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt = \ldots$

(2) $\int_L \frac{c^4 t^2}{c^2 t^2 + \ldots}\ldots\, dt = \frac{2}{3}t^3\Big|_0^{\sqrt2} = \frac{8\sqrt2}{3}a$

(3) OA: $(x,y)=(t,0)$. AB: $(x,y)=(t,1-t)$. BO: $(x,y)=(0,t)$. $\int_0^1 1+t\,dt + \int_0^1\sqrt2 dt + \int_0^1 1+t\, dt = 1+\sqrt2$.

(4) $y=\frac{1}{2}x-2$. $x=t$, $y=\frac{1}{2}t-2$. $\int_0^4 \frac{1}{t-\frac{1}{2}t+2}\sqrt{1+(\frac{1}{2})^2}\,dt = \sqrt5\ln(t+4)\Big|_0^4 = \sqrt5\ln 2$

(5) $\vec{AB}=(0,1,3)=(0,t,0)$, $t\in[0,1]$. $\int_0^1 1+t\,dt=\frac{1}{2}$ … $=2\sqrt2\pi^2$.

(6) $x=a\cos\varphi$, $y=a\sin\varphi$. $ds=|r'(\theta)|\,d\theta = a\,d\theta$ …

(7) $x=r\cos\varphi$, $y=r\sin\varphi$. $r=ae^{k\varphi}$ … $=\frac{a^2}{4k}\sqrt{k^2+1}(e^{2}-1)$.

(8) $\int_L t\sqrt{(\cos t-t\sin t)^2+(\sin t+t\cos t)^2+1}\,dt = \frac{1}{3}(t^2+2)^{\frac{3}{2}}\Big|_0^{t_0}$ …

(9) $x=r\cos\varphi$, $y=r\sin\varphi$ …

(10) $x=a\cos\varphi$, $y=a\sin\varphi$. $\int_L (x^2+y^2)^n ds = \int_0^{2\pi} a^{2n}\times a\,d\varphi = 2\pi a^{2n+1}$

(11) $x=a\sin\theta\cos\varphi$, $y=a\sin\theta\sin\varphi$, $z=a\cos\theta$ …

(12) $\int (c^2\sin^2\theta\cos\varphi\sin\varphi + \ldots)\, a\sin\theta\, d\theta$

4. $I=\frac{1}{2}mr^2=\frac{1}{2}\int\rho ds\, r^2 = \frac{1}{2}\int ds\, r^2$. $ds = \sqrt{a^2+\frac{h^2}{4\pi^2}}\,dt$ …

3.(6) $x=a\cos\varphi$, $y=a\sin\varphi$ … $\Rightarrow 2\int_0^a e^t dt = 2(e^a-1)$.

(10) $x=a\cos\varphi$, $y=a\sin\varphi$, $z=0$. $\int_0^{2\pi} a^{2n}\times a\,d\varphi = 2\pi a^{2n+1}$.

(11) 对称. $\int_L x^2 ds = \frac{1}{3}\int_L x^2+y^2+z^2\,ds = \frac{1}{3}\int a^2 ds = \frac{a^2}{3}\int ds = \frac{a^2}{3}\times 2\pi a$?

4. $I=\frac{1}{2}\int ds\, r^2$. $ds=\sqrt{a^2+\frac{h^2}{4\pi^2}}\,dt$. 对 x, $r^2=y^2+z^2$ … 对 y … 对 z, $r^2=x^2+y^2=a^2$ …

习题 11.2

1.(1) $z=\sqrt{x^2+y^2}$. $z_x'=\frac{x}{\sqrt{x^2+y^2}}$, $z_y'=\frac{y}{\sqrt{x^2+y^2}}$. $S=\iint_D\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,d\sigma=\sqrt2\times\pi$.

(2) $x=b\cos\varphi$, $y=a\sin\varphi$, $z=z$ … $=c^2\int_0^{\frac{\pi}{2}}\cos\varphi+\sin\varphi\,d\varphi = c^2\times2 = 2a^2$

(3) $x=a\cos\varphi$, $y=c\sin\varphi$, $z=z$. $\sqrt{EG-F^2}=a$ … $2\int_0^{2\pi}\int_0^{c\cos\varphi} a\,dz\,d\varphi$ … ?

(4) $x=\sqrt{a}\sin\theta\cos\varphi$, $y=\sqrt{a}\sin\theta\sin\varphi$, $z=\sqrt{a}\cos\theta$ … 球面 $\int_0^{2\pi}d\varphi\int_0^{\arccos\frac{\sqrt3}{3}}d\theta\,3a^2\sin\theta = 6\pi a^2(1-\frac{\sqrt3}{3})$.

抛物面. $\cos\theta=\frac{\sqrt3}{3}$. $z=\frac{1}{2a}(x^2+y^2)$ … $=\frac{2\pi}{3}(3\sqrt3-1)a^2$ … 算错.

(5) $x=\frac{1}{2}(y^2+z^2)$. $\frac{\partial x}{\partial y}=y$, $\frac{\partial x}{\partial z}=z$. $\iint\sqrt{1+y^2+z^2}\,dy\,dz = \int_0^{2\pi}\int_0^1\sqrt{1+r^2}\,r\,dr\,d\varphi = \frac{2\pi}{3}(2^{\frac{3}{2}}-1)$

(6) $z=\sqrt{x^2+y^2}$ … $\iint\sqrt2\,dx\,dy$, D: $2\sqrt2\pi$ … $=16\sqrt2\pi$.

(7) $\vec r_\rho=(\cos\varphi,\sin\varphi,0)$, $\vec r_\varphi=(-r\sin\varphi, r\cos\varphi, h)$ … $=2\pi\int_0^R\sqrt{r^2+h^2}\,dr$ …

(8) $x=a\sin\theta\cos\varphi$, $y=a\sin\theta\sin\varphi$, $z=a\cos\theta$ …
$E=a^2\sin 2\varphi$, … $\sqrt{EG-F^2}=a^2\sin\theta$

注：可以由曲面分解成不定积分 … 都有定义 $\varphi\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3\pi}{2}]$

2.(1) x,y,z 对称. $\iint_S(x+y+z)\,ds = 3\iint_S x\,ds$ … 上下: $\frac{1}{2}$. 左右: $\frac{1}{2}$. $3\times\frac{3}{2}=\frac{9}{2}$.

(2) $z=1-x-y$. $z_x'=-1$, $z_y'=-1$ … $=-\frac{\sqrt3}{120}$

(3) $z=1$, $x^2+y^2=1$ … $=\frac{\sqrt2+1}{2}\pi$

(4) $x^2-2ax+z^2+y^2=0$ … $=\frac{16\sqrt2}{15}a^4$.

(5) $\iint_S(x^2-y^4+(y^2-z^2)(x^2+y^2)+1)\,ds = \iint 1\,ds = \pi$.

(6) $\iint_S\frac{1}{r^2}dS$. $r^2=x^2+y^2+z^2=R^2+z^2$ … $=2\pi\arctan\frac{H}{R}$

(7) $4\iint_S xyz\,dS$, $z=x^2+y^2$ … $=\frac{4}{5}\times\frac{1}{2}\times1=\frac{2}{5}$.

3.(1) $8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}R^2\sin\theta\,R^2\sin\theta\,d\theta\,d\varphi = \frac{8\pi}{3}R^4$

(2) $3\iint x\,ds = \ldots = 3\pi a^3$.

4. $\sigma S' = d\hat x\,d\hat y$ … $\frac{S}{S_1}=\sqrt{\frac{A^2+B^2+C^2}{C^2}}$

1.(4) $x^2+y^2+z^2=3a^2$, $x^2+y^2=2az$. 球: $x=\sqrt3a\sin\theta\cos\varphi$ … $=6\pi a^2(1-\frac{\sqrt3}{3})$. 抛: $z=\frac{1}{2a}(x^2+y^2)$ …

习题 11.3

1.(1) $\int_L(x^2+y^2)dx+(x^2-y^2)dy$. L_1: $x\in[0,1]$, $y=x$ … $=\frac{2}{3}$. L_2: $x\in[1,2]$, $y=-x+2$ … $\therefore$ 积分为 $\frac{4}{3}$.

(2) $\int_L\frac{1}{|x|+|y|}dx+\frac{1}{|x|+|y|}dy$ … 为 0.

(3) $\int_L\frac{-x}{x^2+y^2}dx+\frac{y}{x^2+y^2}dy$, $x=a\cos\theta$, $y=a\sin\theta$.

(4) $\int y^2dx+xy\,dy+xz\,dz$ … $\Rightarrow 1$.

(5) $\int e^{x+y+z}dx+e^{x+y+z}dy+e^{x+y+z}dz$ … $=e^{3}-1$

(6) $x+y=2$, $x^2+y^2+z^2=2(x+y)$ … $=4-2\sqrt2\pi$.

2. $\int_L\vec v\cdot d\vec r = \int_L(y+z)\vec i+(z+x)\vec j+(x+y)\vec k\cdot d\vec r$ … $=2\pi a^2$.

3. $F=-\frac{k}{r^3}\vec r$. $x=a\cos\theta$, $y=b\sin\theta$ …

$\vec F=(a\cos\theta, b\sin\theta)$; $d\vec x = -a\sin\theta\,d\theta$; $\vec F\cdot d\vec x = -a^2\sin\theta\cos\theta\,d\theta$

4.(1) $\oint_D P\,dx+Q\,dy = \iint_D\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy$. $P=(x+y)^2$, $Q=(x^2-y^2)$ … $=\frac{16}{3}$.

(2) $P=xy+x+y$, $Q=xy+x-y$ … $=\frac{4}{3}ab\cdot(b-a)$

(3) $P=yx^3+e^y$, $Q=xy^3+xe^y-2y$ … $\iint 0\,dx\,dy = 0$

(4) $P=\sqrt{x^2+y^2}$, $Q=y[xy+\ln(x+\sqrt{x^2+y^2})]$ …

(5) $P=x^2+xy-y^2$, $Q=x^2-xy+y^2$ … $=\frac{2}{3}$.

(6) $P=e^x\sin y-my$, $Q=e^x\cos y-m$ …

5.(1) $\iint 1\,dx\,dy = \int_0^{2\pi}a\sin^3t\times a3\cos^2t(-\sin t)\,dt$ … $=\frac{3}{8}\pi a^2$

7.(1) $\frac{\partial f}{\partial n}=\nabla f$, $\Delta f=\nabla\cdot\nabla f$ …

散度定理的 2 维形式.

(1) $X=(-Q,P)$ 用 Green. $\oint\vec X\cdot d\vec l=\iint\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}dx\,dy$. $\vec X'=(P,Q)$. $\vec X\cdot d\vec l=\vec X'\cdot\vec n$. $\iint_D\Delta f\,dx\,dy=\iint(\nabla\cdot\nabla f)dx\,dy = \oint\nabla f\cdot\vec n\,ds=\oint\frac{\partial f}{\partial n}ds$.

(2) $\oint\cos(\vec a,\vec n)\,ds = \frac{1}{|\vec a|}\oint\vec a\cdot\vec n\,ds = \iint \mathrm{Div}(\vec a)\,dx\,dy$.

(3) 按分量. $vu_{ii}-uv_{ii} = (vu_i)_i-(uv_i)_i = (vu_i-uv_i)_i$; $\therefore\iint vu-u\Delta v\,d\sigma=\oint(vu_i-uv_i)n_i = \oint v\nabla u\cdot\vec n-u\nabla v\cdot\vec n$

1.(3) $\int_L e^{x+y+z}dx+e^{x+y+z}dy+e^{x+y+z}dz$. L: $x=\cos\varphi$, $y=\sin\varphi$, $z=\frac{\varphi}{\pi}$ … $=\int_1^{\frac{3}{2}}e^t dt = e^{\frac{3}{2}}-e$ ✓

2. $\vec v=(y+z)\vec i+(z+x)\vec j+(x+y)\vec k$. 设 L: $x=a\sin^2t$, $y=2a\sin t\cos t$, $z=a\cos^2t$ …

3. $F=kr\vec e_r$. $W=\int\vec F\cdot d\vec r=\int kr\cdot d\vec r$ … $F=k\frac{\vec r}{r^3}$, $W=\int\vec F\cdot d\vec r = \int k\frac{\vec r}{r^3}\cdot d\vec r$

4.(2) $\oint(xy+x+y)dx+(xy+x-y)dy$, L: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 正向 … $=-\frac{1}{2}(b\pi a+\pi a)\times ab$ …

(4) $\oint\sqrt{x^2+y^2}dx+y[xy+\ln(x+\sqrt{x^2+y^2})]dy$, L: $x=2$ 与 $y^2=x-1$ … $=\frac{4}{15}$ ✓

(6) $\int(e^x\sin y-my)dx+(e^x\cos y-m)dy$. $x^2+y^2=ax$ … $=\frac{\pi a^2 m}{8}$

3.(1) $x=a\cos^3t$, $y=a\sin^3t$ … $=\frac{3}{8}\pi a^2$ ✓

(2) $x=a(t-\sin t)$, $y=a(1-\cos t)$ … $=a^2\cdot 3\pi$ ✓

习题 11.4

1.(1) $\vec n=\vec r_\theta'\times\vec r_\varphi'$. $\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$, $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ … $x=a\sin\theta\cos\varphi$, $y=b\sin\theta\sin\varphi$, $z=c\cos\theta$ … $=8abc\cdot\ldots = \frac{4}{3}\pi abc\ldots$

(2) $x=R\cos\theta$, $z=R\sin\theta$, $y=y$ … $=\int_0^{2\pi}d\theta\int_0^h dy\times0 = 0$.

(3) $x\le0$. $x=R\sin\theta\cos\varphi$, $y=R\sin\theta\sin\varphi$, $z=R\cos\theta$ … $=\frac{4\pi}{105}R^7$

(4) $z=\sqrt{1-x^2-y^2}$ … $=\frac{\pi}{8}$ ✓

(5) $\vec n=(1,1,1)\to\frac{1}{\sqrt3}(1,1,1)$. $\iint_S\vec v\cdot\vec n\,dS$ … 这里S不应该是 $\{x+y\le1, x\ge0, y\ge0\}$?? … $=\frac{1}{4}$.

(6) $z=\sqrt{x^2+y^2}$ … $=0$

(7) $z=\sqrt{a^2-x^2-y^2}$ … $=\frac{2\pi}{105}a^7$

(8) $\iint f(x)dy\,dz+g(y)dz\,dx+h(z)dx\,dy$ … $=bcf(a)+acg(b)+abh(c)-f(0)\cdot g(0)\cdot h(0)$

2. $\iint_S(x^2-yz)\vec i+(-2x^2y)\vec j+z\vec k\;d\vec S$ … $=a^2bc-\frac{1}{2}b^2c^2-\frac{2}{3}a^2bc$ … 总 $=abc-\frac{1}{2}b^2c^2+\frac{1}{3}a^3bc$

1.(2) $\iint xyz\,dx\,dy$, $x^2+z^2=R^2$ 在 $y=0$, $y=h$, $x\ge0$, $y\ge0$ … $=\frac{1}{3}R^3h^2$ ✓

(3) $\iint(x+y^2+z)dx\,dy$. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ …

(5) $\iint x^2dy\,dz+y^2dz\,dx+z^2dx\,dy$, $x+y+z=1$ …

2. $\vec v=(x^2-yz,-2x^2y,z)$ … 总: $\frac{1}{3}a^3bc+abc$ ✓

习题 11.5

1.(1) $P=x+1$, $Q=y$, $R=xy+z$. $\iiint 1+1+1\,dx\,dy\,dz = 3\times\frac{1}{6}=\frac{1}{2}$.

(2) $P=xy$, $Q=yz$, $R=xz$ … $=\frac{1}{4}$.

(3) $P=x^2$, $Q=y^2$, $R=z^2$ … $=\frac{8\pi}{3}(a+b+c)R^3$.

(4) $P=xy^2$, $Q=yz^2$, $R=zx^2$. $\iiint y^2+z^2+x^2\,dx\,dy\,dz = \frac{4}{5}\pi R^5$

(5) $P=x-z$, $Q=y-x$, $R=z-y$ … $=\frac{3\sqrt3-1}{2}\pi$.

(6) $P=y^2+z^2$, $Q=x^2+z^2$, $R=x^2+y^2$ … $=-\frac{\pi a^4}{2}$.

2.(1) 的 $\vec F\cdot d\vec S=m$. (2) 0. (3) ?

3. 挖去 以原点为心小球 ε …

4. $P=xf(x)$, $Q=-xyf(x)$, $R=-e^{2x}z$ … $f(x)=e^{x}\ldots$

9.(1) $\oint y\,dx+z\,dy+x\,dz$ … $=-\sqrt3\pi a^2$

(2) $\oint(y-z)dx+(z-x)dy+(x-y)dz$ … $=-2\pi a(h+a)$. 注：$\vec n$ 应该要是单位向量吧，不然 dS 与 $\vec n\,dS$ 不等 (?)

(3) $\oint(y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz$ … $=-\frac{9}{2}a^3$

(4) … $=-4\sqrt3$.

(5) … $=3\iint dS = 3\times2\times\frac{\pi}{3}=2\pi$.

(6) … $=-\frac{1}{3}$.

6. S 为有 $\vec r(u,v)$, $\vec n=\vec r_u'\times\vec r_v'$. $\iint_S\cos(\vec c,\vec n)\,dS=\oint$ ……

12. $\oint y^2+z^2dx+(z^2+x^2)dy+(x^2+y^2)dz$ …

1.(2) $\iint xy\,dy\,dz+yz\,dz\,dx+zx\,dx\,dy$. $z=1-x-y$ … $=\frac{1}{24}$ … $\iiint(x+y+z)dx\,dy\,dz = \int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}(x+y+z)dz$ … $=\frac{1}{8}$.

习题 11.7

1.(1) L_1: $x\in[0,1]$, $y=0$, $dy=0$. $\int\vec F\cdot d\vec l=\int_0^1 -y\,dx = 0$. $y\in[0,1]$, $x=1$, $dx=0$. $\int\vec F\cdot d\vec l=\int_0^1 x\,dy=1$.

L_2: $x\in[0,1]$, $y=x^2$ … $=\frac{1}{3}$. L_3: … L_4: … $\nabla\times\vec F = 1+1=2\ne0$. 非保守.

2.

3.(1) $\nabla\times\vec v$ … $=0$. $\varphi = x^2\cos y+y^2\sin x$. grad φ = ✓.

(2) $\nabla\times\vec v$ … $\varphi=x^2yz+xy^2z+xyz^2$ …

(3) … $\therefore\varphi=\frac{1}{4}(x^4+y^4+z^4)+\frac{1}{2}(x^2y^2+y^2z^2+z^2x^2)$

4.

5.(1) $\frac{\partial u}{\partial x}=3x^2+6xy^2$, $u=x^3+3x^2y^2+f(y)$ … $\therefore u=x^3+3x^2y^2-y^4$.

(2) $\frac{\partial u}{\partial x}=x^2-2yz$ … $\Rightarrow u=\frac{1}{3}(x^3+y^3+z^3)-2xyz$.

6.(1) $\vec v=(x-y)dx-(x-y)dy$. $\nabla\times\vec v = -1-(-1)=0$. … $\varphi=\frac{1}{2}(x-y)^2$ …

(2) … $=\frac{3}{2}$.

(3) $\varphi=\sqrt{x^2+y^2}$, $\int_{(1,0)}^{(1,1)}=\sqrt2-1$.

(4) … $\int_{(1,0,0)}^{(2,3,-4)} = 11-64=-53$.

(5) $\varphi_x=x-\frac{x}{y}+\frac{1}{2}x$ … (6) $\varphi=\sqrt{x^2+y^2+z^2}$ …

9. $\nabla\times\vec v = (0,0,f''(x)-f'(x)+bf(x)+e^{-2x})$. 设 $f''(x)-f'(x)+bf(x)+e^{-2x}=0$. $f(x)=e^{-2x}(ax^2+bx+c)$ … $F(x)=-\frac{1}{8}e^{-2x}$.

10. 有势场 无旋. $\nabla\times\vec v=0$ … 证 β 之 …

11. $\nabla\times\vec v$ … $\frac{\partial Q}{\partial x}=2x$. $Q=x^2+f(y)$ … $\therefore f(y)=3y^2$. $Q=x^2+3y^2$?

12.(1) … $\therefore\frac{1}{2}x^2y^2+xy-y\sin x-y\cos x=C$.

(2) … $\varphi(x,y)=\frac{y^3}{x^2}+\frac{1}{3}\frac{y^3}{x^3}-\frac{y}{x}$, $\ldots=C$.

13. $\nabla\times\vec v$ … $f(x)=e^x-e^{-x}-x^2$. $\varphi(x,y)=e^x\sin y+e^y\ldots=C$.