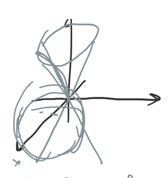
补充题

2022年4月16日 星期六 上午11:03

习指导

第一型曲线与曲面.

91. 

L_1: $\vec{OA}=(1,2,3)$.
L_2: $x=4y+3z^2$, $z^2=\frac{4}{5}x^2+y^2$.

$x=r\sin\theta\cos\varphi$. $y=r\sin\theta\sin\varphi$. $z=r\cos\theta$.
$x=r\cos\theta$. $y=r\sin\theta\sin\varphi$. $z=r\sin\theta\cos\varphi$.
$r\cos\theta=r^2\sin^2\theta(1+2\sin\varphi\cos\varphi)$.
$\frac{4}{5}r^2\cos^2\theta=r^2\sin^2\theta(\cos^2\varphi-\sin^2\varphi)$

对 L_1: $x=x, y=2x, z=3x, x\in[0,1]$. ⇒ 参数.
$\int_{L_1} z\,ds=\int_0^1 3x\sqrt{1^2+2^2+3^2}\,dx=\frac{3}{2}\sqrt{14}$.

对 L_2: 令 $y+z=t$, $x=t^2$,
$\frac{4}{5}t^4=t(z-y)$, $y=\frac{1}{2}(t-\frac{4}{5}t^3)$, $z=\frac{1}{2}(t+\frac{4}{5}t^3)$.
$\int_{L_2} z\,ds=\int_0^1\frac{1}{2}(t+\frac{4}{5}t^3)\sqrt{\cdots}\,dt$.

92. L: $x^2+y^2=4x$. $\int\sqrt{x^2+y^2}\,ds$.

$(x-2)^2+y^2=4$. $x=r\cos\theta$, $y=r\sin\theta$. 极坐标表示.
$r^2=4r\cos\theta$, $r=4\cos\theta$. ⇒ $x=4\cos^2\theta$. $y=4\cos\theta\sin\theta$.
$x'(\theta)=-8\cos\theta\sin\theta=-4\sin2\theta$, $y'(\theta)=-4\sin^2\theta+4\cos^2\theta=4\cos2\theta$
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}4\cos\theta\sqrt{(4\sin2\theta)^2+(4\cos2\theta)^2}\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}16\cos\theta\,d\theta=16\times2=32$ ✓

参数: $x=2+2\cos t$, $y=2\sin t$.
显式: $y=\sqrt{4x-x^2}$, $y'=\frac{2-x}{y}$.

93. $(x^2+y^2)^2=a^2(x^2-y^2)$ $\int(x+y)\,ds$.

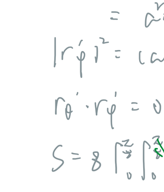
$x=r\cos\theta$. $y=r\sin\theta$.
$r^4=a^2r^2(\cos^2\theta-\sin^2\theta)$, $r^2=a^2(\cos^2\theta-\sin^2\theta)$.
取 $r>0$. $r=a\sqrt{\cos^2\theta-\sin^2\theta}$. (右半) $x=a\sqrt{\cos^2\theta-\sin^2\theta}\cos\theta=a\sqrt{\cos2\theta}\cos\theta$, $y=a\sqrt{\cos2\theta}\sin\theta$
$x'(\theta)=a\left(\frac{-2\sin2\theta\cos\theta}{2\sqrt{\cos2\theta}}-\sqrt{\cos2\theta}\sin\theta\right)=-a\left(\frac{\sin2\theta\cos\theta}{\sqrt{\cos2\theta}}+\sqrt{\cos2\theta}\sin\theta\right)$
$y'(\theta)=a\left(\frac{-2\sin2\theta\sin\theta}{2\sqrt{\cos2\theta}}+\sqrt{\cos2\theta}\cos\theta\right)=-a\left(\frac{\sin2\theta\sin\theta}{\sqrt{\cos2\theta}}-\sqrt{\cos2\theta}\cos\theta\right)$
$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a\sqrt{\cos2\theta}(\cos\theta+\sin\theta)\times\frac{a}{\sqrt{\cos2\theta}}\sqrt{(\sin2\theta\cos\theta+\cos2\theta\sin\theta)^2+(\sin2\theta\sin\theta-\cos2\theta\cos\theta)^2}\,d\theta$
$=a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta+\sin\theta)\times1\,d\theta$
$=a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta+\sin\theta)=2a^2$

$r^2=a^2\cos2\theta$, $\cos2\theta\ge0$, $\theta\in[-\frac{\pi}{4},\frac{\pi}{4}]$.
y 关于 y 奇函数, $\int_L y\,ds=0$

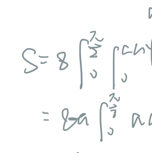
94. $\int_L x^2y\,ds$. $x^2+y^2+z^2=R^2$. $x+y+z=2R$.

$x=t$. $y+z=2R-t$. $y^2+z^2+2yz=R^2+t^2-2Rt$.
$y^2+z^2=R^2-t^2$. $yz=t^2-Rt$.
$y+\frac{t^2-Rt}{y}=2R-t$. $y^2-(2R-t)y+t(t-R)=0$

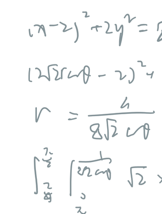
轮换对称.
$\int x^2ds=\int y^2ds=\int z^2ds$. $\int x\,ds=\int y\,ds=\int z\,ds$.
$\therefore\int x^2y\,ds=\frac{1}{3}\int x^2+y^2+z^2+\frac{1}{3}\int x+y+z=\frac{1}{3}(R^2+2R)\int_L ds$.

95. 

$x^2+y^2=a^2$. $y^2+z^2=a^2$.
$x=a\sin\theta\cos\varphi$. $z=a\sin\theta\sin\varphi$. $y=a\cos\theta$.
$\sin^2\theta\cos^2\varphi+\cos^2\theta=1$, $\sin\theta\cos\varphi=\sin\theta$. $\cos\varphi=1$
$\sin^2\theta\sin^2\varphi+\cos^2\theta=1$. $\sin^2\theta\sin^2\varphi=\sin^2\theta$. $\sin\varphi=1$.
$|r'_\theta|^2=(a\cos\varphi\cos\theta)^2+(a\sin\varphi\cos\theta)^2+(a\sin\theta)^2$
$=a^2\sin^2\theta+a^2\cos^2\theta=a^2$.
$|r'_\varphi|^2=(a\sin\theta\sin\varphi)^2+(a\sin\theta\cos\varphi)^2+0^2=a^2\sin^2\theta$.
$r'_\theta\cdot r'_\varphi=0$.
$S=8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{a^2a^2\sin^2\theta}\,d\theta d\varphi=4\pi\int_0^{\frac{\pi}{2}}a^2\sin\theta\,d\theta=4\pi a^2$.
$0\le z\le a\cos\varphi$.



$x=a\sin\theta\cos\varphi$. $y=a\sin\theta\sin\varphi$. $z=a\cos\theta$.
$a^2\sin^2\theta\sin^2\varphi+a^2\cos^2\theta=a^2$.
$\sin^2\theta\sin^2\varphi+\cos^2\theta=1$. $\sin^2\varphi=1$.
$x=a\cos\varphi$. $y=a\sin\varphi$. $z=z$.
$a^2\sin^2\varphi+z^2=a^2$, $z^2=a^2\cos^2\varphi$. $z=a\cos\varphi$.
$S=8\int_0^{\frac{\pi}{2}}\int_0^{a\cos\varphi}a\,dz\,d\varphi$
$=8a\int_0^{\frac{\pi}{2}}a\cos\varphi\,d\varphi=8a^2$.

96. 

$z=\sqrt{x^2+y^2}$. $\frac{x}{\sqrt{x^2+y^2}}$
$2(\frac{x^2}{4}+x+1)=x^2+y^2=\frac{x^2}{2}+2x+2$
$\frac{x^2}{2}-2x-2+y^2=0$. $x^2-4x+4+2y^2=8$.
$(x-2)^2+2y^2=8$ $x=2\sqrt{2}r\cos\theta$, $\sqrt{2}y=2\sqrt{2}r\sin\theta$.
$(2\sqrt{2}r\cos\theta-2)^2+(2\sqrt{2}r\sin\theta)^2=8$. $-8\sqrt{2}r\cos\theta+4+8=8$.
$r=\frac{4}{8\sqrt{2}\cos\theta}=\frac{1}{2\sqrt{2}\cos\theta}$. $\theta\in[\frac{\pi}{3},\frac{\pi}{3}]$
$\int_{\frac{\pi}{3}}^{\frac{\pi}{3}}\int_0^{\frac{1}{2\sqrt{2}\cos\theta}}\sqrt{2}\times r\,dr\,d\theta$
$=\int_{\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sqrt{2}}{2}r^2\Big|_0^{\frac{1}{2\sqrt{2}\cos\theta}}d\theta=\int_{\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{\sqrt{2}}{2}\times\frac{1}{8\cos^2\theta}d\theta$.
$\iint_D\sqrt{2}\,dxdy=\sqrt{2}\times\pi\times2\sqrt{2}\times2=8\pi$ 椭圆面积.
$z=\sqrt{2}(\frac{x}{2}+1)$ $\frac{\sqrt{2}}{2}, 0, 1$ $\iint\sqrt{1+\frac{1}{2}}\,dxdy$
$=\sqrt{\frac{3}{2}}\times\pi\times2\sqrt{2}\times2=4\sqrt{3}\pi$

97. (1) $(x^2+y^2)^{\frac{1}{3}}+z^{\frac{2}{3}}=1$

$(x^2+y^2)^{\frac{1}{3}}=\sin^2\theta$, $x^2+y^2=\sin^6\theta$. $x=\sin^3\theta\cos\varphi$, $y=\sin^3\theta\sin\varphi$
$z^{\frac{2}{3}}=\cos^2\theta$, $z=\cos^3\theta$.
$\vec{r}'_\theta=(3\cos\varphi\sin^2\theta\cos\theta, 3\sin\varphi\sin^2\theta\cos\theta, -3\cos^2\theta\sin\theta)$
$\vec{r}'_\varphi=(-\sin^3\theta\sin\varphi, \sin^3\theta\cos\varphi, 0)$
$E=(3\sin^2\theta\cos\theta)^2+(3\cos^2\theta\sin\theta)^2=9\sin^4\theta\cos^2\theta+9\cos^4\theta\sin^2\theta$
$=9\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)=9\sin^2\theta\cos^2\theta$.
$G=\sin^6\theta\sin^2\varphi+\sin^6\theta\cos^2\varphi=\sin^6\theta$.
$F=-3\sin^5\theta\cos\theta\cos\varphi\sin\varphi+3\sin^5\theta\cos\theta\cos\varphi\sin\varphi=0$
$8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{9\sin^8\theta\cos^2\theta}\,d\theta d\varphi=8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}3\sin^4\theta\cos\theta\,d\theta d\varphi$
$=8\times\frac{\pi}{2}\times3\int_0^{\frac{\pi}{2}}\sin^4\theta\,d\sin\theta=12\pi\times\frac{1}{5}\sin^5\theta\Big|_0^{\frac{\pi}{2}}=\frac{12}{5}\pi$.

(2) $(x^2+y^2+z^2)^2=2a^2xy$.

$x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$. $z=r\cos\theta$.
$r^4=2a^2r^2\sin^2\theta\sin\varphi\cos\varphi$, $r^2=2a^2\sin^2\theta\sin\varphi\cos\varphi$.
$r=a\sin\theta\sqrt{\sin2\varphi}$.
$x=a\sin^2\theta\sqrt{\sin2\varphi}\cos\varphi$. $y=a\sin^2\theta\sqrt{\sin2\varphi}\sin\varphi$. $z=a\sin\theta\cos\theta\sqrt{\sin2\varphi}$.
$\vec{r}'_\theta=(a\sqrt{\sin2\varphi}\cos\varphi\,2\sin\theta\cos\theta, a\sqrt{\sin2\varphi}\sin\varphi\,2\sin\theta\cos\theta, a\sqrt{\sin2\varphi}(\cos^2\theta-\sin^2\theta))$
$=(a\sqrt{\sin2\varphi}\cos\varphi\sin2\theta, a\sqrt{\sin2\varphi}\sin\varphi\sin2\theta, a\sqrt{\sin2\varphi}\cos2\theta)$
$\vec{r}'_\varphi=\left(a\sin^2\theta\left(\frac{2\cos2\varphi\cos\varphi}{2\sqrt{\sin2\varphi}}-\sqrt{\sin2\varphi}\sin\varphi\right), a\sin^2\theta\left(\frac{\cos2\varphi\sin\varphi}{\sqrt{\sin2\varphi}}+\sqrt{\sin2\varphi}\cos\varphi\right), a\sin\theta\cos\theta\frac{\cos2\varphi}{\sqrt{\sin2\varphi}}\right)$
$=\left(a\sin^2\theta\frac{\cos2\varphi\cos\varphi-\sin2\varphi\sin\varphi}{\sqrt{\sin2\varphi}}, a\sin^2\theta\frac{\cos2\varphi\sin\varphi+\sin2\varphi\cos\varphi}{\sqrt{\sin2\varphi}}, a\sin\theta\cos\theta\frac{\cos2\varphi}{\sqrt{\sin2\varphi}}\right)$
$E=a^2\sin2\varphi$ ✓, $G=\frac{a^2\sin^4\theta}{\sin2\varphi}+\frac{a^2\sin^2\theta\cos^2\theta}{\sin2\varphi}\cos^22\varphi$
$=\frac{a^2\sin^2\theta}{\sin2\varphi}(\sin^2\theta+\cos^2\theta\cos^22\varphi)$ ✓
$F=a^2\sin^2\theta\cos3\varphi\cos\varphi\sin2\theta+a^2\sin^2\theta\sin3\varphi\sin\varphi\sin2\theta+a^2\sin\theta\cos\theta\cos2\theta\cos2\varphi$
$=a^2\sin^2\theta\sin2\theta\cos2\varphi+a^2\sin\theta\cos\theta\cos2\theta\cos2\varphi$
$=a^2\sin\theta\cos2\varphi(\sin\theta\sin2\theta+\cos\theta\cos2\theta)$ $\sqrt{EG-F^2}=a^2\sin\theta$
$=a^2\sin\theta\cos\theta\cos2\varphi=\frac{a^2}{2}\sin2\theta\cos2\varphi$.
$8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\sqrt{a^4\sin^2\theta(\sin^2\theta+\cos^2\theta\cos^22\varphi)-\frac{a^4}{4}\sin^22\theta\cos^22\varphi}\,d\theta d\varphi$
$=8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}a^2\sqrt{\sin^4\theta+\sin^2\theta\cos^2\theta\cos^22\varphi-\frac{1}{4}\sin^22\theta\cos^22\varphi}\,d\theta d\varphi$.
$\sqrt{\sin^4\theta+\frac{1}{4}\cos^22\varphi(\sin^22\theta-\cos^22\theta)}$ $1-2\sin^2\theta=\cos2\theta$, $\frac{1-\cos2\theta}{2}=\sin^2\theta$, $\frac{1}{4}(1-\cos2\theta)^2=\sin^4\theta$
$=\frac{1}{2}\sqrt{(1-\cos2\theta)^2+\cos^22\varphi(\sin^22\theta-\cos^22\theta)}$
答. 限定 $4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}$

98. $x^2+y^2+z^2=2z$. $\iint_S x^2dS$.

$x^2+y^2+(z-1)^2=1$.
$x=\sin\theta\cos\varphi$. $y=\sin\theta\sin\varphi$. $z=\cos\theta+1$.
$8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\varphi\sin\theta\,d\theta d\varphi$
$=8\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta\int_0^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi=8\times\frac{2}{3}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{4}{3}\pi$.

99. (1) $\iint_S x^4-y^4+y^2z^2-x^2z^2+1\,dS$.

$x=r\cos\varphi$. $y=r\sin\varphi$. $z=z$.
$r^4\cos^4\varphi-r^4\sin^4\varphi+(r^2\sin^2\varphi-r^2\cos^2\varphi)z+1$
$=(r^2\cos^2\varphi-r^2\sin^2\varphi)(r^2\cos^2\varphi+r^2\sin^2\varphi+z)+1$
$=r^2\cos2\varphi(r^2+z)+1=r^2\cos2\varphi(r^2+r)+1$
$z=r$. $r^2=2r\cos\theta$. $r=2\cos\theta$
$\int_0^{2\cos\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}r^4(r+1)\cos2\varphi+r\,d\theta\,dr$
$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos2\varphi\,d\varphi\int_0^{2\cos\theta}r^5+r^4dr+\int_0^{2\cos\theta}r\,d\theta\,dr$
$=$

第二型曲线

107. ✓ $I=\oint(y+1)dx+(z+2)dy+(x+3)dz$.
L为 $x^2+y^2+z^2=a^2$ 与 $x+y+z=0$ 的交线.

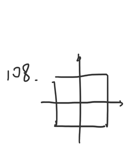
$z=-x-y$. $x^2+y^2+x^2+2xy+y^2=a^2$.
$2x^2+2y^2+2xy=a^2$
$r^2\cos^2\theta+r^2\sin^2\theta+r^2\sin\theta\cos\theta=\frac{1}{2}a^2$
$r^2+\frac{1}{2}r^2\sin2\theta=\frac{1}{2}a^2$. $r^2(1+\frac{1}{2}\sin2\theta)=\frac{1}{2}a^2$
$r^2=\frac{a^2}{2+\sin2\theta}$. $r=\frac{a}{\sqrt{2+\sin2\theta}}$
$x=\frac{a\cos\theta}{\sqrt{2+\sin2\theta}}$, $y=\frac{a\sin\theta}{\sqrt{2+\sin2\theta}}$, $z=\frac{-a(\cos\theta+\sin\theta)}{\sqrt{2+\sin2\theta}}$
$I=\int_0^{2\pi}\left(\frac{a\sin\theta}{\sqrt{2+\sin2\theta}}+1\right)\times(-a\sin\theta\cdot\frac{1}{\sqrt{2+\cos\theta}}+a\cos\theta\times\frac{\cdots}{(2+\sin2\theta)^{\frac{3}{2}}}$

106. $\int_L y\,dx+z\,dy+x\,dz$

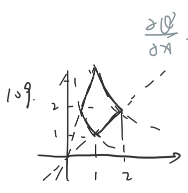
L: $\begin{cases}x+y=2\\x^2+y^2+z^2=2(x+y)\end{cases}$.

$x^2+y^2+z^2=4$ $y=2-x$. $-x(x-2)\ge0$, $x(x-2)\le0$. $0\sim2$
$x^2+x^2-4x+4+z^2=4$ $z^2=4x-2x^2\ge0$
$2\int_2^0(2-x)dx-\sqrt{4x-2x^2}dx+x\times\frac{4-4x}{2\sqrt{4x-2x^2}}dx$
$=2\int_2^0(2-x)-\sqrt{4x-2x^2}+\frac{2x(1-x)}{\sqrt{4x-2x^2}}dx$

消 y. $2(x-1)^2+z^2=2$. 投影为椭圆.
$(\frac{z}{\sqrt{2}})^2+(x-1)^2=1$.
$z=\sqrt{2}\cos t$. $x-1=\sin t$. $y=2-x=1-\sin t$.

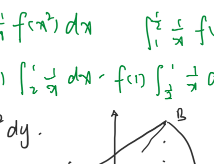
108. 

$\oint xe^{xy}dy-ye^{-xy}dx=\oint xe^{-xy}-ye^{xy}dx$
$Q\,dy-P\,dx$ $Q'dy-P'dx$
$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=e^{xy}+ye^{xy}x$ $\iint$
$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=e^{xy}-ye^{xy}x$ $\iint$

109. 

$y=4x$. $y=x$. $xy=1$. $xy=4$ 围.
$f(1)=1$. $f(4)=4$. 求 $\oint\frac{1}{y}f(xy)dy$.

$\oint\frac{1}{y}f(xy)dy=\int_{\frac{1}{4}}^{1}f(4x^2)\cdot4dx+\int_1^2\frac{1}{x}f(x^2)dx$
$+\int_2^1\frac{x}{4}f(4)d\frac{4}{x}+\int_{\frac{1}{2}}^1 xf(1)d\frac{1}{x}$
$=\int_1^2\frac{1}{x}f(x^2)+\int_2^1-\frac{1}{x}f(4)+\int_1^{\frac{1}{2}}\frac{1}{x}f(4x^2)dx+\int_{\frac{1}{2}}^1-\frac{1}{x}f(1)dx$
$\int_1^2\frac{1}{x}f(x^2)dx$ $\int_1^{\frac{1}{2}}\frac{1}{x}f(4x^2)dx$
$-f(4)\int_2^1\frac{1}{x}dx-f(1)\int_{\frac{1}{2}}^1\frac{1}{x}dx=$

113. $\int_L x^2dy$. 

AB. l: $y=\frac{4-1}{3-(-2)}(x+2)+1=\frac{3}{5}x+\frac{11}{5}$.
$x\in[-2,3]$, $y=\frac{3}{5}x+\frac{11}{5}$.
$\int_{-2}^3 x^2\cdot\frac{3}{5}dx=\frac{1}{5}x^3\Big|_{-2}^3=\frac{1}{5}(27+8)=7$
BC. $x=5\cos\theta$, $y=5\sin\theta$.
$\int_0^{\cdots}25\cos^2\theta\cdot5\,d\sin\theta=\int_0^{\frac{4}{5}}125(1-\sin^2\theta)d\sin\theta$ $\frac{75-16}{75}=\frac{59}{75}$
$=125\int_0^{\frac{4}{5}}1-t^2dt=125\left(\frac{4}{5}-\frac{1}{3}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\right)$
$=100\left(1-\frac{1}{3}\times\frac{16}{25}\right)=100\times\frac{59}{75}=\frac{236}{3}$

106. $\int_L y\,dx+z\,dy+x\,dz$. L: $\begin{cases}x+y=2\\x^2+y^2+z^2=2(x+y)\end{cases}$.

$x=r\sin\theta\cos\varphi$. $y=r\sin\theta\sin\varphi$. $z=r\cos\theta$.
$x+y=r\sin\theta(\cos\varphi+\sin\varphi)=2$. $r=2$.
$\sin\theta(\cos\varphi+\sin\varphi)=1$.
消去 y, 在 Oxz 平面上投影 $(\frac{z}{\sqrt{2}})^2+(x-1)^2=1$.
$z=\sqrt{2}\cos\theta$, $x=\sin\theta+1$. $y=2-x=1-\sin\theta$.
原式 $=\int_{2\pi}^0(1-\sin t)\cos t+\sqrt{2}\cos t(-\cos t)+(1+\sin t)(\sqrt{2}\sin t)\,dt$

第二型曲面

116. $\iint_S x\,dydz+y\,dzdx+z\,dxdy$ $x^2+y^2+z^2=R^2$.

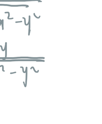
S1 $x=R\sin\theta\cos\varphi$ $y=R\sin\theta\sin\varphi$. $z=R\cos\theta$.
$\vec{r}'_\theta=(R\cos\varphi\cos\theta, R\sin\varphi\cos\theta, -R\sin\theta)$
$\vec{r}'_\varphi=(-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0)$.
$\iint_S\begin{vmatrix}x & y & z\\ R\cos\varphi\cos\theta & R\sin\varphi\cos\theta & -R\sin\theta\\ -R\sin\theta\sin\varphi & R\sin\theta\cos\varphi & 0\end{vmatrix}d\varphi d\theta$ ✓
$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}R^3\sin^3\theta\cos\varphi+R^3\sin\theta\cos\varphi+R^3\sin\theta\cos\theta\cdots\,d\varphi d\theta$
$R^3\sin\theta\,d\theta$.
$=R^3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^3\theta(\sin\varphi+\cos\varphi)+\sin\theta\cos\theta\,d\varphi\,d\theta$ $r\,d\theta\,d\theta$
$=R^3\int_0^{2\pi}(\sin\varphi+\cos\varphi)\frac{2}{3\times1}+\frac{1}{3}d\varphi$
$=R^3\times\frac{2\pi}{3}=\frac{2\pi R^3}{3}$ $2\pi R^3$.

S2 $z=\sqrt{R^2-x^2-y^2}$ $z'_x=-\frac{x}{\sqrt{R^2-x^2-y^2}}$ $z'_y=-\frac{y}{\sqrt{R^2-x^2-y^2}}$
$\iint_D\begin{vmatrix}x & y & z\\1 & 0 & z'_x\\0 & 1 & z'_y\end{vmatrix}dxdy$ 外, 有正, $z=\sqrt{R^2-x^2-y^2}$.
$=\iint_D\frac{x^2}{\sqrt{R^2-x^2-y^2}}+\frac{y^2}{\sqrt{R^2-x^2-y^2}}+\sqrt{R^2-x^2-y^2}\,dxdy$

S3 $\iint_S\vec{v}\cdot\vec{n}\,dS=\iint(x,y,z)\cdot(\frac{x}{R},\frac{y}{R},\frac{z}{R})dS$
$=\frac{1}{R}\iint_S x^2+y^2+z^2\,dS$
$=\frac{1}{R}\times R^2\times\frac{1}{2}\times4\pi R^2=2\pi R^3$

117. $\iint_S y^2dzdx+z\,dxdy$. $\frac{x^2}{4}+\frac{y^2}{4}+z^2=1$.

S1 $\vec{v}=(0,y^2,z)=(0,4\sin^2\theta\sin^2\varphi,\cos\theta)$.
$x=2\sin\theta\cos\varphi$, $y=2\sin\theta\sin\varphi$. $z=\cos\theta$.
$\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}d\theta\begin{vmatrix}0 & 4\sin^2\theta\sin^2\varphi & \cos\theta\\2\cos\varphi\cos\theta & 2\sin\varphi\cos\theta & -\sin\theta\\-2\sin\theta\sin\varphi & 2\sin\theta\cos\varphi & 0\end{vmatrix}$
$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}8\sin^4\theta\sin^3\varphi+\cos\theta(4\sin^2\varphi\sin\theta\cos\theta+4\cos^2\varphi\sin\theta\cos\theta)\,d\theta d\varphi$
$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}8\sin^4\theta\sin^3\varphi+4\sin\theta\cos^2\theta\,d\theta d\varphi$ $\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\cos\theta$
$=8\int_0^{2\pi}\sin^3\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\sin^4\theta+4\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\sin\theta\cos^2\theta\,d\theta$
$=8\times4\times\frac{2}{3}\times2\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}+4\times2\pi\times2\times(-\frac{1}{3})\times1$
$=8\pi-\frac{16\pi}{3}=\frac{8}{3}\pi$. $\frac{16}{3}\pi$.

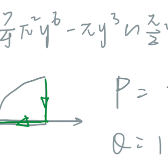
S2. $z=\sqrt{1-\frac{x^2}{4}-\frac{y^2}{4}}$ $\times2$. 向上. $z'_x=-\frac{2x}{\sqrt{4-x^2-y^2}}$
$=\frac{1}{2}\sqrt{4-x^2-y^2}$ $z'_y=-\frac{2y}{\sqrt{4-x^2-y^2}}$
$2\iint_S\begin{vmatrix}0 & y^2 & \frac{1}{2}\sqrt{4-x^2-y^2}\\1 & 0 & -\frac{2x}{\sqrt{4-x^2-y^2}}\\0 & 1 & -\frac{2y}{\sqrt{4-x^2-y^2}}\end{vmatrix}dxdy$ 
$=2\iint_D\frac{2y^3}{\sqrt{4-x^2-y^2}}+\frac{1}{2}\sqrt{4-x^2-y^2}\,dxdy$ $x=r\cos\theta, y=r\sin\theta$
$=2\int_0^2\int_0^{2\pi}\frac{2r^4\sin^3\theta}{\sqrt{4-r^2}}+\frac{1}{2}\sqrt{4-r^2}r\,d\theta\,dr=\frac{16}{3}\pi$

谢惠民

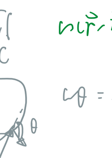
第二型曲线积分.

例. C为抛物线 $2x=\pi y^2$ 自 $(0,0)$ 到 $(\frac{\pi}{2},1)$ 的弧段, 求 $\int_C(2xy^3-y^2\cos x)dx+(1-2y\sin x+3x^2y^2)dy$

S1 $x=\frac{\pi}{2}y^2$. $y\in[0,1]$ $\frac{\pi^2}{4}y^4$
$\int_0^1(\pi y^5-y^2\cos\frac{\pi}{2}y^2)\pi y\,dy+(1-2y\sin\frac{\pi}{2}y^2+\frac{3}{4}\pi^2y^4)dy$
$=\int_0^1\frac{3}{4}\pi^2y^6-\pi y^3\cos\frac{\pi}{2}y^2-2y\sin\frac{\pi}{2}y^2+1\,dy$

S2  $P=2xy^3-y^2\cos x$, $\frac{\partial P}{\partial y}=6xy^2-2y\cos x$.
$Q=1-2y\sin x+3x^2y^2$, $\frac{\partial Q}{\partial x}=-2y\cos x+6xy^2$
$\iint-2y\cos x+6xy^2-6xy^2+2y\cos x=0$. $=\int_{C+L_1+L_2}$
添辅助线. $\int_{L_1}$. $\int_{L_2}$.
L_1. $x\in[\frac{\pi}{2},0]$, $y=0$. $dy=0$
$\int_{L_1}1\,dy=0$.
L_2. $y\in[1,0]$. $x=\frac{\pi}{2}$. $dx=0$
$\int_{L_2}1-2y\sin\frac{\pi}{2}+3\frac{\pi^2}{4}y^2\,dy$
$=\int_1^0 1-2y+\frac{3}{4}\pi^2y^2dy=y-y^2+\frac{\pi^2}{4}y^3\Big|_1^0$
$=-(1+1)\cdot\frac{\pi^2}{4}=\frac{\pi^2}{4}$.

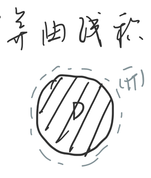
例. $I=\oint_C\frac{\cos(\vec{r},\vec{n})}{|\vec{r}|}ds$.

C为逐段光滑的简单闭曲线, $\vec{r}=(x,y)$.
$r=|\vec{r}|=\sqrt{x^2+y^2}$. $\vec{n}$是C上的单位外法向量
$I=\iint_C$ $\cos(\vec{r},\vec{n})=\frac{\vec{r}\cdot\vec{n}}{r}=\frac{1}{r}(x\cos(\vec{n},x)+y\cos(\vec{n},y))$
 $\cos\theta=\frac{\vec{n}\cdot\vec{r}}{|\vec{n}||\vec{r}|}$
$I=\oint_C(\frac{x}{r^2}\cos(\vec{n},x)+\frac{y}{r^2}\cos(\vec{n},y))ds$
$(0,0)$在C外, $I=\iint_D\frac{\partial(\frac{x}{r^2})}{\partial x}+\frac{\partial(\frac{y}{r^2})}{\partial y}dxdy=0$
$(0,0)$在C内, 正向 2π.

习题课 6.17

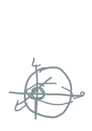
对应 综合 7.8, 11.3-7.

例. 设 $u(x,y)$ 在圆盘 D: $x^2+y^2\le\pi$ 上有二阶连续偏导数, 且 $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=\sin(x^2+y^2)$, $\vec{n}$为 ∂D 圆周 ∂D 的单位外法向量, 计算曲线积分 $I=\oint_{\partial D}\frac{\partial u}{\partial\vec{n}}ds$.

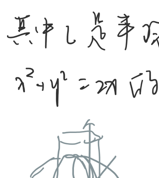


$I=\oint_{\partial D}(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})\cdot\vec{n}\,ds$ $\oint_{\partial D}P\,dx+Q\,dy$
$=\oint_{\partial D}(a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y})ds$ $\oint_{\partial D}(P,Q)\cdot\vec{\tau}\,ds$
$=\oint_{\partial D}(-\frac{\partial u}{\partial y},\frac{\partial u}{\partial x})\cdot(-b,a)\,ds$ $\vec{n}=(a,b)$ $\vec{\tau}=(-b,a)$. $a+ib$, $i(a+ib)=-b+ia$.
$=\oint_{\partial D}-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$
Green↓
$=\iint_D(\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})-\frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}))dxdy$
$=\iint_D\sin(x^2+y^2)dxdy$
$=\int_0^{\sqrt{\pi}}\int_0^{2\pi}\sin r^2\,r\,dr\,d\theta=\pi\int_0^{\pi}\sin\rho\,d\rho=2\pi$

例. 计算曲面积分 $I=\oiint_\Sigma\frac{x\,dydz+y\,dzdx+z\,dxdy}{(x^2+y^2+4z^2)^{\frac{3}{2}}}$
其中 Σ 是曲面 $(x-1)^2+y^2+z^2=a^2$ $(a>0, a\ne1)$ 外侧.

$(x,y,z)\ne(0,0,0)$ 时,
$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\frac{3(x^2+y^2+4z^2)^{\frac{3}{2}}-(x^2+y^2+4z^2)^{\frac{1}{2}}(2x\cdot x\cdot\frac{3}{2}+2y\cdot y\cdot\frac{3}{2}+8z\cdot z\cdot\frac{3}{2})}{(x^2+y^2+4z^2)^3}$
$=\frac{3-(x^2+y^2+4z^2)(x^2+y^2+4z^2)^{\cdots}3}{\cdots}=0$.
当 $0<a<1$ 时, O在Σ外侧.
由 Gauss, $I=0$.
当 $a>1$ 时, O在Σ内部. 
取充分小 $\varepsilon>0$, 使 $\underbrace{x^2+y^2+4z^2=\varepsilon}$ 在Σ内.
$\iint_{\Sigma+\Sigma_\varepsilon^-}=\iiint_{V-V_\varepsilon}0\,dxdydz$. 挖去后的表面积、体积.
$\therefore I=\iint_\Sigma=\iint_{\Sigma_\varepsilon}=\iint_{\Sigma_\varepsilon}\frac{x\,dydz+y\,dzdx+z\,dxdy}{\varepsilon^3}$
Gauss↓
$=\frac{1}{\varepsilon^3}\iiint_{V_\varepsilon}3\,dxdydz=\frac{3}{\varepsilon^3}\cdot\frac{4\pi}{3}\cdot\varepsilon\cdot\varepsilon\cdot\frac{\varepsilon}{2}=2\pi$.

例. 计算曲线积分 $\oint_L(y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$
其中 L 是半球面 $x^2+y^2+z^2=4x$ $(z\ge0)$ 与柱面 $x^2+y^2=2x$ 的交线, 从 z 轴正向看去顺时针.



z轴正向 (上往下看)
投影为圆周、顺.

S1. 设曲面 S 为半球面 $x^2+y^2+z^2=4x$ $(z\ge0)$ 被柱体所截下侧. (从下往上看逆时针)
球面 $(x-2)^2+y^2+z^2=2^2$, $\vec{n}=-(\frac{x-2}{2},\frac{y}{2},\frac{z}{2})$.
设 D: $x^2+y^2\le2x$. $\iint(P,Q,R)\cdot\vec{n}\,dS$ $(dydz, dzdx, dxdy)$
由 Stokes, $I=\iint_S\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\y^2+z^2 & z^2+x^2 & x^2+y^2\\dydz & dzdx & dxdy\end{vmatrix}$
$=-\frac{1}{2}\iint_S\begin{vmatrix}x-2 & y & z\\\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\y^2+z^2 & z^2+x^2 & x^2+y^2\end{vmatrix}dS$
$=-\frac{1}{2}\iint_S[(x-2)(2y-2z)+y(2z-2x)+z(2x-2y)]dS$
$=-2\iint_S(z-y)dS$
对称性 ↖ $=-2\iint_S z\,dS$ 投影, $z=\sqrt{4x-x^2-y^2}$.
$=-2\iint_D\sqrt{4x-x^2-y^2}\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,dxdy$.
$=-2\iint_D\sqrt{4x-x^2-y^2}\sqrt{\frac{4x-x^2-y^2+(2-x)^2+y^2}{4x-x^2-y^2}}\,dxdy$
$=-2\iint_D2\,dxdy=-4\pi$.

S2 L^-: $\begin{cases}x=1+\cos t\\y=\sin t\\z=\sqrt{2+2\cos t}\end{cases}$, $0\le t\le2\pi$.
$=2|\cos\frac{t}{2}|$ → 需要分段 $\langle$ $0\sim\pi$. $\cos\frac{t}{2}$; $\pi\sim2\pi$. $-\cos\frac{t}{2}$.

S3. $\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}$ $r\ge0$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$.
$x^2+y^2=2x$, $r=2\cos\theta$.
$\therefore L^-$: $\begin{cases}x=2\cos^2\theta\\y=2\sin\theta\cos\theta\\z=2\cos\theta\end{cases}$ $z^2=4\cos^2\theta$, $\cos\theta>0$. $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$
$\oint_L(y^2+z^2)dx=-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos^2\theta\sin^2\theta+4\cos^2\theta)2\cos\theta(-\sin\theta)d\theta=0$ 奇.
$\oint_L(x^2+y^2)dz=-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos^4\theta+4\sin^2\theta\cos^2\theta)(-2\sin\theta)d\theta=0$. 奇
$\oint_L(z^2+x^2)dy=-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos^2\theta+4\cos^4\theta)2\cos2\theta\,d\theta$ 偶
$=-16\int_0^{\frac{\pi}{2}}(\cos^2\theta+\cos^4\theta)(2\cos^2\theta-1)d\theta=-4\pi$. (公式)