三重积分

2022年2月2日 星期三 下午7:24

一、概念

$f(x,y,z)$, $(x,y,z)\subset\mathbb{R}^3$.

设V是一个物体，$f(x,y,z)$是密度函数.

$V=V_1\cup V_2\cup V_3\cdots\cup V_n$.

在V_i上，$\sum f(\xi_i,\eta_i,\zeta_i)\Delta V_i\to\iiint f(x,y,z)\,dxdydz$.

二、性质.

三、计算.

1. $V=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$.

$\sum f(P_{ijk})\Delta x_i\Delta y_j\Delta z_k$

$=\sum_{k=1}^{p}\left(\sum f(P_{ijk})\Delta x_i\Delta y_j\right)\Delta z_k$

$=\sum_{k=1}^{p}\varphi(z)\Delta z_k$, $\varphi(z)=\iint_D f(x,y,z)\,dxdy$

$\iiint_V f=\int_{a_3}^{b_3}dz\iint f(x,y,z)\,dxdy$

$=\int_{a_3}^{b_3}dz\int_{a_2}^{b_2}dy\int_{a_1}^{b_1}dx\, f(x,y,z)$.

例. $\iiint_D x^2ye^{xyz}\,dxdydz$, $D=[0,1]^3$.

$=\iint_{[0,1]^2}x^2y\,dxdy\int_0^1 e^{xyz}\,dz$

$=\iint_{[0,1]^2}dxdy\; xe^{xyz}\Big|_0^1$

$=\iint_{[0,1]^2} xe^{xy}-x\,dxdy$

$=\int_0^1dx\int_0^1 xe^{xy}-x\,dy$

$=\int_0^1(e^{xy}-xy)\Big|_0^1dx$

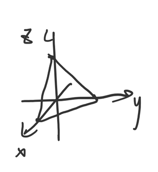
$=\int_0^1(e^x-x-1)\,dx=(e^x-\frac12x^2-x)\Big|_0^1=e-\frac52$.

2. $V=\{(x,y,z)\mid(x,y)\in D\subset\mathbb{R}^2,\ \varphi_1(x,y)\le z\le\varphi_2(x,y)\}$.

$\iiint_V f=\iint_D dxdy\int_{\varphi_1(x,y)}^{\varphi_2(x,y)}f(x,y,z)\,dz$.

例. $\iiint\frac{dxdydz}{(1+x+y+z)^3}$, V: $x\ge0$, $y\ge0$, $z\ge0$, $x+y+z=1$.

$D=\{(x,y)\mid x+y=1, x\ge0, z\ge0\}$.



$\iint_D dxdy\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^3}$

$=\iint_D -\frac12\frac{1}{(1+x+y+z)^2}\Big|_0^{1-x-y}dxdy$

$=-\frac12\iint_D\frac{1}{2^2}-\frac{1}{(1+x+y)^2}\,dxdy$

$=\frac12\int_0^1dx\int_0^x\frac{1}{(1+x+y)^2}-\frac14\,dy$

$=\frac12\int_0^1\left(-\frac{1}{1+x+y}-\frac14y\right)\Big|_0^x dx$

$=\frac12\int_0^1($

四、换元.

$x=x(u,v,w)$, $y=y(u,v,w)$, $z=z(u,v,w)$

$\iiint_V f(x,y,z)\,dxdydz$

$=\iiint_{V'}f(x(u,v,w),\cdots)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw$.

例. $x=r\cos\varphi\sin\theta$, $y=r\sin\varphi\sin\theta$, $z=r\cos\theta$.

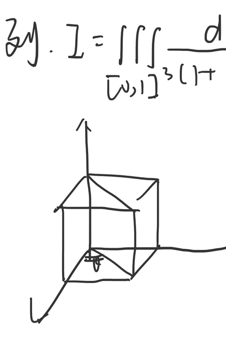
$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\varphi}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\varphi}&\frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial r}&\frac{\partial z}{\partial\varphi}&\frac{\partial z}{\partial\theta}\end{vmatrix}$$

$$=\begin{vmatrix}\cos\varphi\sin\theta&-r\sin\varphi\sin\theta&r\cos\varphi\cos\theta\\ \sin\varphi\sin\theta&r\cos\varphi\sin\theta&r\sin\varphi\cos\theta\\ \cos\theta&0&-r\sin\theta\end{vmatrix}$$

$=|-r^2\cos^2\varphi\sin^3\theta-r^2\sin^2\varphi\cos^2\theta\sin\theta-r^2\sin^2\varphi\sin^3\theta-r^2\cos^2\varphi\cos^2\theta\sin\theta|$

$=|-r^2\sin^3\theta-r^2\cos^2\theta\sin\theta|=r^2\sin\theta$

例. $I=\iiint_{[0,1]^3}\frac{dudvdw}{(1+u^2+v^2+w^2)^2}$



$V=\iiint_V\frac{dudvdw}{(1+u^2+v^2+w^2)^2}$

$u=r\cos\theta$, $v=r\sin\theta$, $w=w$

$=\iiint_{V'}\frac{r\,dr\,d\theta\,dz}{(1+r^2+w^2)^2}$

$V'=\{(r,\theta,w)\mid 0\le\theta\le\frac\pi4,\ 0\le r\le\frac{1}{\cos\theta}=\sec\theta,\ 0\le w\le1\}$

令 $w=\tan\varphi$. $\frac{dw}{d\varphi}=\sec^2\varphi$.

$=\iiint_{V''}\frac{r\sec^2\varphi\,dr\,d\theta\,d\varphi}{(1+r^2+\tan^2\varphi)^2}$

$=\iiint_{V''}\frac{r\sec^2\varphi\,dr\,d\theta\,d\varphi}{(r^2+\sec^2\varphi)^2}$

$V''=\{(r,\theta,\varphi)\mid 0\le\theta\le\frac\pi4,\ 0\le r\le\sec\theta,\ 0\le\varphi\le\frac\pi4\}$.

$=\iint_{[0,\frac\pi4]^2}d\theta d\varphi\int_0^{\sec\theta}\frac{\sec^2\varphi\,r}{(r^2+\sec^2\varphi)^2}dr$

$=-\iint_{[0,\frac\pi4]^2}\frac{\sec^2\varphi}{2(r^2+\sec^2\varphi)}\Big|_0^{\sec\theta}d\theta d\varphi$

$=-\iint_{[0,\frac\pi4]^2}\left(\frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}-1\right)d\theta d\varphi$

$=-\int_0^{\frac\pi4}d\varphi\int_0^{\frac\pi4}\frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}d\theta\ \ \frac{\pi^2}{16}$

$\iint_{[0,\frac\pi4]^2}\frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}d\theta d\varphi=\iint_{[0,\frac\pi4]^2}\frac{\sec^2\theta}{\sec^2\theta+\sec^2\varphi}d\theta d\varphi$.

积分区域对称，和为1. $\iint=\frac12(\frac\pi4)^2$.

∴ 原式 $=\frac12(\frac\pi4)^2=\frac{\pi^2}{32}$.

例. $I=\iiint_V\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz$, V: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le1$

$x=au$, $y=bv$, $z=cw$. $\frac{\partial(x,y,z)}{\partial(u,v,w)}=abc$.

$I=\iiint_{u^2+v^2+w^2\le1}\sqrt{1-u^2-v^2-w^2}\,abc\,dudvdw$

$u=r\cos\varphi\sin\theta$, $v=r\sin\varphi\sin\theta$, $w=r\cos\theta$. $\frac{\partial(u,v,w)}{\partial(r,\theta,\varphi)}=r^2\sin\theta$.

$I=abc\iiint_{0\le r\le1,\ 0\le\theta\le\pi,\ 0\le\varphi\le2\pi}\sqrt{1-r^2}\,r^2\sin\theta\,dr\,d\theta\,d\varphi$ ⟹ 三重积分

$=abc\,2\pi\iint_{0\le r\le1,\ 0\le\theta\le\pi}\sqrt{1-r^2}\,r^2\sin\theta\,dr\,d\theta$

$=2\pi abc\int_0^1dr\int_0^\pi\sqrt{1-r^2}\,r^2\sin\theta\,d\theta$

$=4\pi abc\int_0^1\sqrt{1-r^2}\,r^2\,dr$

$r=\sin t$: $=4\pi abc\int_0^{\frac\pi2}\cos^2t\sin^2t\,dt$ $\quad\frac12\cdot2\pi abc\left(\frac12t-\frac{\sin4t}{8}\right)\Big|_0^{\frac\pi2}$

$=\pi abc\int_0^{\frac\pi2}(\sin2t)^2\,dt$

$=\pi abc\int_0^{\frac\pi2}\frac{1-\cos4t}{2}\,dt=$

例. $f(x,y)$ 连续，证明:

$\left(\int_a^b dx\left(\int_c^d f(x,y)dy\right)^2\right)\le\left(\int_c^d dy\left(\int_a^b f^2(x,y)dx\right)^{\frac12}\right)^2$

S₁ 左 $=\int_a^b dx\left(\int_c^d f(x,y)dy\right)\left(\int_c^d f(x,z)dz\right)$ → 三重积分

$=\iiint_{a\le x\le b,\ c\le y\le d,\ c\le z\le d}f(x,y)f(x,z)\,dxdydz$

$=\int_c^d dz\int_c^d dy\int_a^b f(x,y)f(x,z)\,dx$

$\le\int_c^d dz\int_c^d dy\left(\int_a^b f^2(x,y)dx\right)^{\frac12}\left(\int_a^b f^2(x,z)dx\right)^{\frac12}$ Cauchy

$=\int_c^d dz\left(\int_a^b f^2(x,z)dx\right)^{\frac12}\cdot\int_c^d dy\left(\int_a^b f^2(x,y)dx\right)^{\frac12}$

$=\left[\int_c^d dy\left(\int_a^b f^2(x,y)dx\right)^{\frac12}\right]^2=$ 右.

S₂ 设 $\varphi(d)=\int_a^b dx\left(\int_c^d f(x,y)\,dy\right)^2$ (构造函数求导) 求导.

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1. $V=I_1\times I_2\times I_3=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$

$\qquad T_x\qquad T_y\qquad T_z$.

任取一点 $M_{ijl}\in V_{ijl}$.

$S(T,M)=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{k}f(M_{ijl})\Delta x_i\Delta y_j\Delta z_l$.

$\lim_{T\to0}S(T,M)=A$, $\iiint_V f(x,y,z)\,dxdydz$

2. ① $S(T,M)=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{k}f(\xi_i,\eta_j,\mu_l)\Delta x_i\Delta y_j\Delta z_l$.

$=\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{l=1}^{k}f(\xi_i,\eta_j,\mu_l)\Delta z_l\right)\Delta x_i\Delta y_j$.

$\varphi(\xi_i,\eta_j)=\int_{a_3}^{b_3}f(\xi_i,\eta_j,z)\,dz$.

令 $|T_z|\to0$，得 $\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi(\xi_i,\eta_j)\Delta x_i\Delta y_j$

$\iiint_V f(x,y,z)\,dxdydz=\iint_{I_x\times I_y}\left(\int_{a_3}^{b_3}f(x,y,z)dz\right)dxdy$.

$=\int_{I_z}\left(\iint_{I_x\times I_y}f(x,y,z)\,dxdy\right)dz$.

例. $\iiint_V x^2ye^{xyz}\,dxdydz$. $V\in[0,1]^3$.

$\iiint_V x^2ye^{xy}\cdot e^z\,dz\,dxdy$

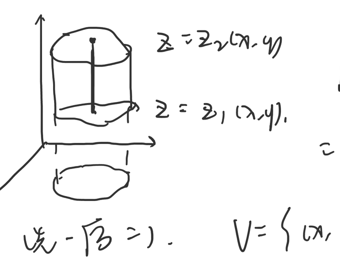
$=\iint_{[0,1]^2}x\cdot xy\,e^{xy}e^z\,dz\,dxdy$

$=\iint_{[0,1]^2}x(e^{xy}-1)\,dxdy$

$=\int_0^1\int_0^1 xe^{xy}\,dxdy-\iint_{[0,1]^2}x\,dxdy$

$=\int_0^1(e^x-1)\,dx-\frac12=e-\frac52$.

②

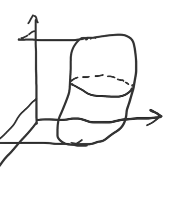


$z=z_2(x,y)$, $z=z_1(x,y)$.

$\iiint_V f(x,y,z)\,dxdydz$

$=\iint_D\left[\int_{z_1}^{z_2}f(x,y,z)\,dz\right]dxdy$.

先一后二). $V=\{(x,y,z)\mid z_1(x,y)\le z\le z_2(x,y)\}$.



$\iiint_V f(x,y,z)\,dxdydz$

$=\int_c^b\left[\iint_{D_z}f(x,y,z)\,dxdy\right]dz$.

例. 计算 $\iiint_V\frac{dxdydz}{(1+x+y+z)^3}$

V $x=0$, $y=0$, $z=0$, $x+y+z=1$ 围成

S₁. $V=\{(x,y,z)\mid x,y,z\ge0,\ x+y+z\le1\}$.

∴ $z\ge0$, $z\le1-x-y$.

原式 $=\iint_D\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^3}\,dxdy$

$=-\frac12\iint_D\frac{1}{(1+x+y+z)^2}\Big|_0^{1-x-y}dxdy$.

$=-\frac12\iint_D\frac{1}{2^2}-\frac{1}{(1+x+y)^2}\,dxdy$

$=-\frac12\int_0^1\int_0^{1-x}\frac14-\frac{1}{(1+x+y)^2}\,dy\,dx$

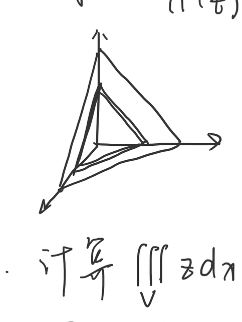
$=-\frac1{16}+\frac12\int_0^1\frac{1}{1+x+y}\Big|_0^x dx$

$=-\frac1{16}+\frac12\int_0^1 ~~\frac{1+x}{(1+2x)}-\frac12\frac{1}{1+x}~~ dx$

~~$=-\frac1{16}+\frac12\ln\frac{|1+2x|}{|1+x|}\Big|_0^1=$~~

$=\frac12\ln2-\frac5{16}$.

S₂. $f(t)=\frac{1}{(1+t)^3}$. $\iiint_V f(x,y,z)\,dxdydz$.



$dV=\frac{\sqrt3}{4}(\sqrt2t)^2\times\frac{dt}{\sqrt3}$.

$\int_0^1 f(t)\,dV=\int_0^1\frac{1}{(1+t)^3}\frac{t^2}{2}\,dt$

$=\ldots$

例. 计算 $\iiint_V z\,dxdydz$，其中V由锥面 $R^2z^2=h^2(x^2+y^2)$ 及平面 $z=h$ 围成的锥体.

S₁ D: $x^2+y^2\le R^2$. $z=\frac hR\sqrt{x^2+y^2}$.

$I=\iint_D\left[\int_{\frac hR\sqrt{x^2+y^2}}^{h}z\,dz\right]dxdy$.

$=\frac12\iint_D\left(h^2-\frac{h^2}{R^2}(x^2+y^2)\right)dxdy$

$=\frac12h^2\pi R^2-\frac{h^2}{2R^2}\iint_D x^2+y^2\,dxdy$ → 换元

$=\frac12h^2\pi R^2-\frac12\times\frac12\frac{h^2}{R^2}\times2\pi R^4=\frac{\pi}{4}R^2h^2$

S₂. $I=\int_0^h z\iint_D x^2+y^2\,dxdy\,dz$ $\qquad x^2+y^2\le\frac{R^2}{h^2}z^2$.

$=\frac12\int_0^h z^2\pi\frac{R^2}{h^2}z^2\,dz$

例. 计算 $\iiint_V(x^2+y^2+z^2)\,dxdydz$. V: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le1$.

被积函数可加性. 椭圆 $\frac{y^2}{b^2}+\frac{z^2}{c^2}\le1-\frac{x^2}{a^2}$.

$\iiint_V x^2\,dxdydz=2\int_0^a x^2\cdot\pi bc\cdot\left(1-\frac{x^2}{a^2}\right)dx$

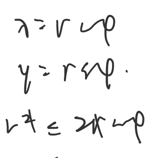
$=\frac{2\pi bc}{a^2}\int_0^a(a^2x^2-x^4)\,dx=\frac{4}{15}\pi a^3bc$.

例. 求 $\iiint_V z\sqrt{x^2+y^2}\,dxdydz$. V: $x^2+y^2=2x$, $z=0$, $z=a$.

$I=\int_0^a z\iint_{x^2+y^2\le2x}\sqrt{x^2+y^2}\,dxdy\,dz$.

$=\int_0^a z\,dz\iint_{x^2+y^2\le2x}\sqrt{x^2+y^2}\,dxdy$ $\qquad x=r\cos\varphi$, $y=r\sin\varphi$. $r^2\le2r\cos\varphi$

$=\frac12a^2\times\int_{-\frac\pi2}^{\frac\pi2}\int_0^{2\cos\varphi}r^2\,dr\,d\varphi$



$=\frac{a^2}{6}\int_{-\frac\pi2}^{\frac\pi2}8\cos^3\varphi\,d\varphi$.

$=\frac{a^2}{6}\int_0^{\frac\pi2}16\cos^3\varphi\,d\varphi=\frac{16}{9}a^2$

例. $\iiint_V\frac{dxdydz}{1+x^2+y^2}$, V: $0\le z\le h$, $x^2+y^2\le4z$.

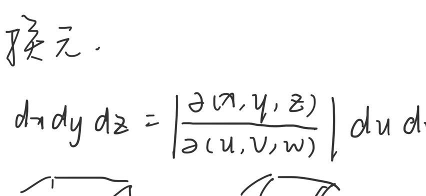
$\int_0^h\iint_{x^2+y^2\le4z}\frac{1}{1+x^2+y^2}\,dxdy\,dz$

$=\int_0^h\left(\int_0^{2\pi}d\varphi\int_0^{2\sqrt z}\frac{r}{1+r^2}dr\right)dz$

$=\pi\int_0^h\ln(1+4z)\,dz=\frac\pi4(\sim\sim)$

3. 换元.

$dxdydz=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw$.



$\vec r_u'\,du$. $\vec r_v'\,dv$. $\vec r_w'\,dw$.

$V'=|\vec r_u'\times\vec r_v'\cdot\vec r_w'|=dudvdw$.

球坐标换元.

$x=r\sin\theta\cos\varphi$, $y=r\cos\theta\cos\varphi$, $z=r\sin\varphi$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=\begin{vmatrix}\sin\theta\cos\varphi&r\cos\theta\cos\varphi&-r\sin\theta\sin\varphi\\ \cos\theta\cos\varphi&-r\sin\theta\cos\varphi&-r\cos\theta\sin\varphi\\ \sin\varphi&0&r\cos\varphi\end{vmatrix}=r^2\sin\theta.$$

柱坐标换元. $dxdydz=r\,dr\,d\theta\,dz$.

例. (1) 求椭球面 $\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1$ 在 $(\sqrt3,\frac{4\sqrt3}{3},\frac{5\sqrt3}{3})$ 的切平面.

(2) 设V是平面π与三个坐标平面围成，求 $\iiint_V x\left(\sqrt3-\frac y4-\frac z5\right)dxdydz$.

(1) $\frac{x}{3\sqrt3}+\frac{y}{4\sqrt3}+\frac{z}{5\sqrt3}\le1$.

(2) $u+v+w\le1$. $u,v,w\ge0$.

$I=\iiint_{V'}3\sqrt3u(\sqrt3-\sqrt3v-\sqrt3w)\times(3\sqrt3\times4\sqrt3\times5\sqrt3)\,dudvdw$

$=(3\sqrt3)^2\times4\sqrt3\times5\sqrt3\times\sqrt3\underbrace{\iiint_{V'}u(1-v-w)\,dudvdw}$.

$\iint_{\substack{v,w\ge0\\ v+w\le1}}\int_0^{1-v-w}u\,du\,(1-v-w)\,dvdw$

$=\iint_{\substack{v,w\ge0\\ v+w\le1}}\frac12(1-v-w)^3\,dv\,dw$ $\qquad a=v$, $b=v+w$

$=\frac12\iint_{0\le a\le b\le1}(1-b)^3\,da\,db$

$=\frac12\int_0^1\int_0^b(1-b)^3\,da\,db$

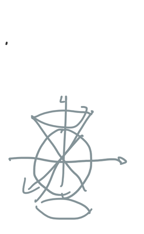
$=\frac12\int_0^1 b(1-b)^3\,db.=\frac1{40}$.

$\int_0^1(1-b)b^3\,db$

$=\frac{81}{2}\sqrt3$.

例. 计算 $\iiint(x^2+y^2+z^2)\,dxdydz$，其中V由 $z=\sqrt{x^2+y^2}$ 与球面 $x^2+y^2+z^2=R^2$ 围成.

$x=r\sin\theta\cos\varphi$. $y=r\sin\theta\sin\varphi$ $z=r\cos\theta$



$r\in[0,R]$. $\theta\in[0,\frac\pi4]$. $\varphi\in[0,2\pi]$.

$\int_0^R\int_0^{\frac\pi4}\int_0^{2\pi}r^4\sin\theta\,d\varphi\,d\theta\,dr$

例. 求曲面 $(x^2+y^2+z^2)^2=a^3z$ 围成体积.

$x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$. $z=r\cos\theta$.

$r^4\le a^3r\cos\theta$. $0\le r\le a\sqrt[3]{\cos\theta}$.

$I=4\int_0^{\frac\pi2}\int_0^{\frac\pi2}\int_0^{a\sqrt[3]{\cos\theta}}r^2\sin\theta\,dr\,d\theta\,d\varphi$ $\qquad -\frac13+\cos\theta$

$=2\pi\int_0^{\frac\pi2}\sin\theta\int_0^{a\sqrt[3]{\cos\theta}}r^2\,dr\,d\theta$

$=2\pi\int_0^{\frac\pi2}\sin\theta\ \frac13a^3\cos\theta\,d\theta$

$=\frac13\pi a^3\int_0^{\frac\pi2}2\sin\theta\,d\sin\theta=\frac{\pi a^3}{3}$.

例. 计算曲面 $(\frac xa)^{\frac23}+(\frac yb)^{\frac23}+(\frac zc)^{\frac23}=1$ 围成体积.

$(\frac xa)^{\frac13}=r\sin\theta\cos\varphi=u$ $\qquad x=au^3$

$(\frac yb)^{\frac13}=r\sin\theta\sin\varphi=v$ $\qquad y=bv^3$

$(\frac zc)^{\frac13}=r\cos\theta=w$. $\qquad z=cw^3$.

$\frac{\partial(x,y,z)}{\partial(u,v,w)}=27abc\,u^2v^2w^2$. $\quad u^2+v^2+w^2=1$

$\iiint_V1\,dxdydz=27abc\iiint_{V'}u^2v^2w^2\,du\,dv\,dw$

$=27abc\int_0^1\int_0^{\pi}\int_0^{2\pi}r^6\sin^4\theta\cos^2\theta\cos^2\varphi\sin^2\varphi\,r^2\sin\theta\,d\theta\,d\varphi\,dr$

$=27abc\int_0^1r^8\,dr\int_0^{\pi}\underline{\sin^5\theta\cos^2\theta}\int_0^{2\pi}\underline{\cos^2\varphi\sin^2\varphi}\,d\varphi$

$=27abc\times\frac19\times$ $\qquad \sin^5\theta(1-\sin^2\theta)$ $\qquad \sin^2\varphi(1-\sin^2\varphi)$

$\qquad \sin^5\theta-\sin^7\theta$ $\qquad \sin^2\varphi-\sin^4\varphi$

例. 已知 $V=\{(x,y,z)\mid x,y,z>0,\ x+y+z<1\}$.

计算 $A=\iiint_V\sin\frac{z}{x+y+z}\,dxdydz$.

$\begin{cases}x+y+z=u.\\ \frac{y+z}{x+y+z}=v\\ \frac{z}{y+z}=w\end{cases}$ $\qquad y+z=\frac zw=uv$, $z=uvw$

$y=uv-uvw=uv(1-w)$

$x=u-uvw-uv(1-w)=u(1-v)$

$0<u<1$, $0<v<1$, $0<w<1$.

$0<z<y+z<x+y+z<1$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}1-v&-u&0\\ v(1-w)&u(1-w)&-w\\ vw&uw&uv\end{vmatrix}=\begin{vmatrix}1-v&-u&0\\ v&u&0\\ uw&uw&uv\end{vmatrix}=u^2v$$

原式 $=\int_0^1\int_0^1\int_0^1\sin\frac{uvw}{u}\times u^2v\,du\,dv\,dw$

$=\int_0^1\int_0^1\int_0^1\sin vw\times u^2v\,du\,dv\,dw$

$=\frac13\int_0^1\int_0^1\sin wv\cdot v\,dv\,dw$

$=\frac13\int_0^1\left(-\frac1v\cos vw\right)\Big|_0^1v\,dv$

$=\frac13\int_0^11-\cos v\,dv=\frac13-\frac13\sin1$.

例 (练19). $F(t)=\iint_{[0,t]^2}f(xy)\,dx\,dy$.

求证: $F'(t)=\frac2t\left[F(t)+\iint_{[0,t]^2}xyf'(xy)\,dxdy\right]$.

$F'(t)=\frac2t\int_0^{t^2}f(s)\,ds$.

(2) 记 $\varphi(t)=\int_0^t f(\tau)\,d\tau$.

先证 $F(t)=\int_0^{t^2}\frac{\varphi(\tau)}{\tau}d\tau$. $\left(F'(t)=\frac{\varphi(t^2)}{t^2}\times2t=\frac2t\int_0^{t^2}f(\tau)d\tau\right)$

$F(t)=\int_0^t\left[\int_0^t f(xy)\,dy\right]dx$ $\qquad xy=u$.

$=\int_0^t\frac1x\int_0^{tx}f(u)\,du\,dx=\int_0^t\frac{\varphi(tx)}{tx}\,dtx$

$=\int_0^{t^2}\frac{\varphi(\tau)}{\tau}\,d\tau$.

∴ $F'(t)=\frac{\varphi(t^2)}{t^2}2t=\frac2t\int_0^{t^2}f(\tau)\,d\tau$

再证(1) $F'(t)-\frac2tF(t)=\frac2t\int_0^t\int_0^t xyf'(xy)\,dxdy$.

$\frac2t\int_0^t\int_0^t xyf'(xy)\,dxdy=\frac2t\int_0^t\frac1x\int_0^{tx}uf'(u)\,du\,dx$.

$=\frac2t\int_0^t\frac1x\int_0^{tx}u\,df(u)\,dx$.

$=\frac2t\int_0^{tx}\frac1x\left(uf(u)\Big|_0^{tx}-\int_0^{tx}f(u)\,du\right)dx$

$=\frac2t\int_0^t\frac{txf(tx)-\varphi(tx)}{tx}\,dtx$

$=\frac2t\int_0^{t^2}\frac{vf(v)-\varphi(v)}{v}\,dv=\frac2t\int_0^{t^2}f(v)\,dv-\frac2t\frac{\varphi(v)}{v}dv$

$=F'(t)-\frac2tF(t)$. $\qquad \varphi(t^2)$.

* $\left(\int_a^t f(x)\,dx\right)'=f(t)$，但二重用前未知.

$x=tu$, $y=tv$. $\quad F(t)=\int_0^t\int_0^t f(xy)\,dx\,dy$ $\qquad$ 含参多重积分

$=t^2\int_0^1\int_0^1f(t^2uv)\,dudv$ $\qquad$ 将区域去参.

$F'(t)=2t\int_0^1\int_0^1f(t^2uv)\,du\,dv+t^2\int_0^1\int_0^1uvf'(t^2uv)\,2t\,du\,dv$ $\quad\frac{\partial f}{\partial t}$.

$=\frac2t\left[F(t)+\int_0^1\int_0^1tu\cdot tv\,f(t^2uv)\,t\,dtu\,dtv\right]$

$=\frac2t\left[F(t)+\iint_{[0,t]^2}xyf'(xy)\,dxdy\right]$

* 引理: 设 $f(x,y,t)$ 和 $\frac{\partial f(x,y,t)}{\partial t}$ 在 $[a,b]\times[c,d]\times[t_1,t_2]$ 上连续，

令 $F(t)=\int_a^b\int_c^d f(x,y,t)\,dxdy$，

则 $F'(t)=\int_a^b\int_c^d\frac{\partial f}{\partial t}(x,y,t)\,dxdy$.

证明: 令 $G(t)=\int_a^b\int_c^d\frac{\partial f}{\partial t}(x,y,t)\,dxdy$.

连续，$\frac{\partial f(x,y,t)}{\partial t}$ 可积，$G(t)$ 存在.

$\int_{t_1}^tG(\tau)\,d\tau=\int_a^b\int_c^d\underline{\int_{t_1}^t\frac{\partial f(x,y,\tau)}{\partial\tau}d\tau}\,dxdy$

N-L公式 $=\int_a^b\int_c^d\underline{f(x,y,t)-f(x,y,t_1)}\,dxdy$.

$=F(t)-F(t_1)$

∴ $F'(t)=G(t)$

例 (练17). 证明: $\iint_{[0,1]^2}(xy)^{xy}\,dxdy=\int_0^1t^t\,dt$.

* 0^0 补充定义为1. $\lim_{t\to0^+}t\ln t=0$, $\lim_{t\to0^+}t^t=1$

$\int_0^1\int_0^1(xy)^{xy}\,dy\,dx$ $\qquad u=xy$.

$=\int_0^1\int_0^1u^u\,d\frac ux\,dx$

$=\int_0^1\frac1x\int_0^xu^u\,du\,dx$

$=\int_0^1t^t\int_t^1\frac1x\,dx\,dt$. $\qquad$ D: $0\le t\le x\le1$.

$=-\int_0^1t^t\ln t\,dt=-\int_0^1e^{t\ln t}\ln t\,dt$

$=-\int_0^1e^{t\ln t}[(\ln t+1)-1]\,dt$ $\qquad (t\ln t)'=\ln t+1$

$=-\int_0^1e^{t\ln t}\,dt\ln t+\int_0^1t^t\,dt$

只需证 $-\int_0^1e^{t\ln t}\,dt\ln t=0$. $=-\int_0^0e^m\,dm=0$.

$t=1$, $t\ln t=0$. $t=0$, $t\ln t=0$.