n重积分

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一、计算.

1. 设 $I = I_1 \times I_2 \times \cdots \times I_n$. $I_i \in [a_i, b_i]$.

$(x_1, \cdots x_n) \in I \Leftrightarrow a_i \le x_i \le b_i$.

$\int \cdots \int f(x_1, \cdots x_n)\, dx_1 \cdots dx_n$

$= \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_n}^{b_n} dx_n f(x_1, \cdots x_n)$

2. $\iint_{I_n} \cdots \int f = \iint_{I_{n-1}} \cdots \int dx_1 \cdots dx_{n-1} \int_{\varphi_1(x_1, \cdots x_{n-1})}^{\varphi_2(x_1, \cdots x_{n-1})} f(x_1, \cdots x_n)\, dx_n$

3. 换元.

设 $x_i = x_i(u_1, \cdots u_n)$

$\int\int_V f(x_1, \cdots x_n) = \int\int_{V'} f(u_1, \cdots u_n) \frac{\partial(x_1, \cdots x_n)}{\partial(u_1, \cdots u_n)} du_1 \cdots du_n$.

二、例题.

1. n 维单形的体积.

$S_n(a) = \{(x_1, x_2, \cdots, x_n) \mid x_i \ge 0,\ x_1 + \cdots + x_n \le a\}$.

$\mu(S_n(a)) = \int_{S_n(a)} \cdots \int 1\, dx_1 \cdots dx_n$　　换元, $\frac{\partial(x_1 \cdots x_n)}{\partial(t_1 \cdots t_n)} = a^n$.

$\xlongequal[t_i \ge 0, \sum t \le 1]{x_i = a t_i} a^n \int \cdots \int 1\, dt_1 \cdots dt_n = a^n \mu(S_n(1))$

$\mu(S_n(1)) = \int_{t_i \ge 0, t_1 + \cdots + t_n \le 1} \cdots \int 1\, dx_1 \cdots dx_n$

$= \int_{t_1 + \cdots + t_{n-1} \le 1 - t_n,\ 0 \le t_n \le 1} \cdots \int 1\, dx_1 \cdots dx_n$

$= \int_0^1 dt_n \int_{0 \le t_1 + \cdots + t_{n-1} \le 1 - t_n} \cdots \int 1\, dt_1 \cdots dt_{n-1}$ → n-1 维. → 边长为 $1-t_n$

$= \int_0^1 dt_n\, \mu(S_{n-1}(1 - t_n))$　　$\int (1-t_n)^{n-1} dt_n$

$= \int_0^1 dt_n (1-t_n)^{n-1} \mu(S_{n-1}(1))$　　$= \int t_n^{n-1} dt_n$

$= \mu(S_{n-1}(1)) \int_0^1 dt_n (1-t_n)^{n-1}$　　$= \frac{1}{n} t_n^n$.

$= \frac{1}{n} \mu(S_{n-1}(1))$

$\sigma(S_n(1))$

$= \int_0^1 du_n \int 1\, du_1 \cdots du_{n-1}$

$= \int_0^1 (1-u_n)^{n-1} \sigma(S_{n-1}(1))\, du_n$

$= \frac{\sigma(S_{n-1}(1))}{n}$

$\therefore \sigma(S_n(1)) = \frac{1}{n!}$.

$\mu(S_n(1)) = \frac{1}{n!} \mu(S_1(1)) = \frac{1}{n!}$

$\mu(S_n(a)) = \frac{a^n}{n!}$.

2. n 维球的体积.

$B_n(a) = \{(x_1, \cdots x_n) \mid x_1^2 + \cdots + x_n^2 \le a^2\}$.

$\mu(B_n(a)) = \int_{B_n(a)} \cdots \int dx_1 \cdots dx_n$

$= a^n \int_{B_n(1)} \cdots \int dt_1 \cdots dt_n = a^n \mu(B_n(1))$

$\mu(B_n(1)) = \int_{x_1^2 + \cdots + x_n^2 \le 1} \cdots \int dx_1 \cdots dx_n$　　$n \ge 3$ 时　固定 $t_{n-1} = u, t_n = v$.

$= \int_{x_1^2 + \cdots + x_{n-2}^2 \le 1 - x_n^2 - x_{n-1}^2} \cdots \int dx_1 \cdots dx_n \Rightarrow x_n^2 + x_{n-1}^2 \le 1$.

$= \iint_{x_n^2 + x_{n-1}^2 \le 1} dx_n dx_{n-1} \int_{x_1^2 + \cdots + x_{n-2}^2 \le 1 - x_n^2 - x_{n-1}^2} \cdots \int dx_1 \cdots dx_{n-2}$

$= \iint_{x_n^2 + x_{n-1}^2 \le 1} dx_n dx_{n-1}\, \mu(B_{n-2}(\sqrt{1 - x_{n-1}^2 - x_n^2}))$

$= \iint_{x_n^2 + x_{n-1}^2 \le 1} dx_n dx_{n-1} (1 - x_n^2 - x_{n-1}^2)^{\frac{n-2}{2}} \mu(B_{n-2}(1))$

$= \iint_{x_n^2 + x_{n-1}^2 \le 1} (1 - x_n^2 - x_{n-1}^2)^{\frac{n-2}{2}} dx_n dx_{n-1}\, \mu(B_{n-2}(1))$

$\Downarrow$

$\int_0^1 dr \int_0^{2\pi} d\theta (1-r^2)^{\frac{n-2}{2}} r = \int_0^1 (1-r^2)^{\frac{n-2}{n}} r\, dr$

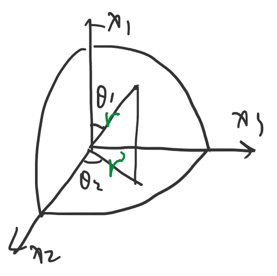
$= \frac{2\pi}{n} \mu(B_{n-2}(1))$

$\mu(B_{2n}(1)) = \frac{\pi^n}{n!}$, $\mu(B_{2n-1}(1)) = \frac{2^n \pi^{n-1}}{(2n-1)!}$

再乘上半径 a^n.

三、n 维球坐标.

1. 三维球坐标.



$x_1 = r\cos\theta_1$,

$x_2 = r'\cos\theta_2 = r\cos\theta_2 \sin\theta_1$,

$x_3 = r'\sin\theta_2 = r\sin\theta_2 \sin\theta_1$,

$\frac{\partial(x_1, x_2, x_3)}{\partial(r, \theta_1, \theta_2)} = r^2 \sin\theta_1$

2. 四维.

$\begin{pmatrix} r \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \to \begin{pmatrix} u_1 = r\cos\theta_1 \\ u_2 = r\sin\theta_1 \\ u_3 = \theta_2 \\ u_4 = \theta_3 \end{pmatrix} \to \begin{pmatrix} x_1 = u_1 \\ x_2 = u_2 \cos\theta_2 \\ x_3 = u_2 \sin\theta_2 \sin\theta_3 \\ x_4 = u_2 \sin\theta_2 \sin\theta_3 \end{pmatrix}$

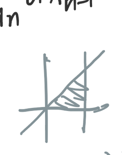
即 $\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1 \cos\theta_2 \\ x_3 = r\sin\theta_1 \sin\theta_2 \sin\theta_3 \\ x_4 = r\sin\theta_1 \sin\theta_2 \cos\theta_3 \end{cases}$

$J_n = r^{n-1} \sin^{n-1}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-1}$

例. 设 $f(x_1, \cdots x_n)$ 连续, 证明:

$\int_a^b dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n f(x_1, \cdots x_n)$

$= \int_a^b dx_n \int_{x_n}^b dx_{n-1} \cdots \int_{x_1}^b dx_1 f(x_1, \cdots x_n)$

证明: $n = 2$.  $\int_a^b dx \int_a^x f(x, y)\, dy = \int_a^b dy \int_x^b dx$.

左边积分区域 $b \ge x_1 \ge x_2 \ge \cdots \ge x_n \ge a$.

假设 n-1 时成立.

设 $D_{x_1} = \{(x_2, \cdots x_n) \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge a\}$.

$\int_c^{x_1} dx_2 \int_c^{x_2} dx_3 \cdots \int_c^{x_{n-1}} dx_n f(x_1, \cdots x_n)$

$= \int_c^{x_1} dx_n \int_{x_n}^{x_1} dx_{n-1} \cdots \int_{x_3}^{x_1} dx_2 f(x_1, \cdots x_n)$

$\int_a^b dx_1 \int_c^{x_1} dx_2 \int_c^{x_2} dx_3 \cdots \int_c^{x_{n-1}} dx_n f(x_1, \cdots x_n)$

$= \int_c^b dx_1 \int_c^{x_1} dx_n \int_{x_n}^{x_1} dx_{n-1} \cdots \int_{x_3}^{x_1} dx_2 f(x_1, \cdots x_n)$　交换

$= \int_c^b dx_n \int_{x_n}^b dx_1 \int_{x_n}^{x_1} dx_{n-1} \cdots \int_{x_3}^{x_1} dx_2 f(x_1, \cdots x_n)$.

$= \cdots =$ 右边.

例. (综合 T11). 设 $a > 0$, $\Omega_n(a): x_1 + \cdots + x_n \le a$, $x_i \ge 0$. 求 $I_n = \int_{\Omega_n(a)} \cdots \int x_1 \cdots x_n\, dx_1 \cdots dx_n$

换元: $x_i = a t_i$.

则 $I_n(a) = a^{2n} I_n(1)$.

$n = 1$, $I_1(1) = \int_0^1 x_1\, dx_1 = \frac{1}{2}$.

$n \ge 2$ 时, $I_n(1) = \int_0^1 x_n\, dx_n\, I_{n-1}(1 - x_n)$

$= I_{n-1}(1) \int_0^1 x_n (1 - x_n)^{2n-2} dx_n$

$= I_{n-1}(1)\, B(2, 2n-1)$

$= I_{n-1}(1) \frac{1!\,(2n-2)!}{(2n)!} = I_{n-1}(1) \cdot \frac{1}{2n(2n-1)}$.

$\therefore I_n(1) = \frac{1}{(2n)!}$, $I_n(a) = \frac{a^{2n}}{(2n)!}$

例. (综合 T12) $f(x_1, \cdots x_n)$ 为 n 元连续函数.

证明 $\int_a^b dx_1 \int_c^{x_1} dx_2 \cdots \int_c^{x_{n-1}} f(x_1, \cdots x_n)\, dx_n$

$= \int_a^b dx_n \int_{x_n}^b dx_{n-1} \cdots \int_{x_2}^b f(x_1 \cdots x_n)\, dx_1$.

区域 $V = \{(x_1, \cdots x_n) \mid a \le x_n \le \cdots \le x_2 \le x_1 \le b\}$.

$\iiint_V f(x_1, \cdots x_n)\, dx_1 \cdots dx_n$

层层推 ← $1 \sim x_{n-1}$ 取定, 积 x_n.

$= \int_c^b dx_1 \int_c^{x_2} dx_2 \cdots \int_c^{x_{n-1}} f(x_1 \cdots x_n)\, dx_n$

$= \int_c^b dx_n \int_{x_n}^b dx_{n-1} \cdots \int_{x_2}^b f(x_1 \cdots x_n)\, dx_1$　　$x_2 \sim x_n$ 取定.

例. (综合 9 T9) 设 $f(x, y) \in C(\mathbb{R}^2)$ ↘ (将条件 $f \in C^1(\mathbb{R}^2)$ 可减弱) $f(x, y)$ 在 $(0, 0)$ 可微, $f(0, 0) = 0$,

证明: 存在 $\mathbb{R}^2$ 上的连续函数 g_1, g_2 使

$f(x, y) = x g_1(x, y) + y g_2(x, y)$

可微: $f(x, y) = ax + by + R(x, y)$, $R(x, y) = o(\rho)$.
　　　　　　$\downarrow g_1$　$\downarrow g_2$

取 $g_1(x, y) = \begin{cases} a + \frac{R(x, y) x}{x^2 + y^2}, & (x, y) \ne (0, 0) \\ a, & (x, y) = (0, 0). \end{cases}$

$\Rightarrow$ $1 = x \cdot \frac{x}{x^2 + y^2} + y \cdot \frac{y}{x^2 + y^2}$. $\Downarrow$ 对应构造.

$g_2(x, y) = \begin{cases} b + \frac{R(x, y) y}{x^2 + y^2}, & (x, y) \ne (0, 0) \\ b, & (x, y) = (0, 0). \end{cases}$

易验证对 $\forall (x, y)$, $f(x, y) = g_1(x) + g_2(x, y)$

f 连续 ⇒ g_1. g_2 在 $(x, y) \ne (0, 0)$ 处连续. (证明连续.)

(趋向 (0,0).) $\lim_{(x, y) \to (0, 0)} g_1(x, y) = \lim_{\rho \to 0} a + \frac{R(x, y)}{\rho} \times \frac{x}{\rho} = a = g_1(0, 0)$.

$\therefore g_1(x, y)$ 在 $(0, 0)$ 连续, 同理 $g_2(x, y)$ 连续.

例. 设 $p(x)$, $f(x)$, $g(x)$ 在 $[a, b]$ 上连续, 且 $p(x) > 0$, $f(x)$. $g(x)$ 单增. 求证: $(b-a) \int_a^b p(x) f(x) g(x)\, dx \cdot \int_a^b p(x)\, dx \ge \int_a^b p(x) f(x)\, dx \int_a^b p(x) g(x)\, dx$. (切比雪夫积分形式)

证明: 当 $x, y \in [a, b]$ 时, $(f(x) - f(y))(g(x) - g(y)) \ge 0$.

$\therefore f(x) g(x) + f(y) g(y) - f(x) g(y) - f(y) g(x) \ge 0$. ($\int_a^b 1\, dy$)

积分得 $2(b-a) \int_a^b f(x) g(x)\, dx \ge 2 \int_a^b f(x)\, dx \int_a^b g(x)\, dx$

增加为 $p(x) p(y) (f(x) - f(y)) (g(x) - g(y)) \ge 0$.

积分得 $2(b-a) \int_a^b p(x) f(x) g(x)\, dx \cdot \int_a^b p(x)\, dx$

$- 2 \int_a^b p(x) f(x)\, dx \int_a^b p(x) g(x)\, dx$.

例. (综合 10 T8) $a, b \ne (0, 0)$ 时, 证明:

$\iint_{x^2 + y^2 \le 1} f(ax + by + c)\, dx\, dy = 2 \int_{-1}^1 \sqrt{1 - t^2} f(t\sqrt{a^2 + b^2} + 1)\, dt$

证明: 取正交阵 $A = \begin{pmatrix} \frac{a}{\sqrt{a^2 + b^2}} & \frac{b}{\sqrt{a^2 + b^2}} \\ * & * \end{pmatrix}$　　$\begin{matrix} a & b \\ -b & a \end{matrix}$.　$u = \frac{ax + by}{\sqrt{a^2 + b^2}}$

正交变换.

$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, $\frac{\partial(x, y)}{\partial(u, v)} = 1$.

$\iint_{x^2 + y^2 \le 1} f(ax + by + c)\, dx\, dy = \iint_{u^2 + v^2 \le 1} f(\sqrt{a^2 + b^2}\, u + c)\, du\, dv$.

$u^2 + v^2 = (u, v) \begin{pmatrix} u \\ v \end{pmatrix}$

$= (x, y) A^T A \begin{pmatrix} x \\ y \end{pmatrix}$

$= x^2 + y^2$

$= 2 \int_{-1}^1 \sqrt{1 - u^2} f(\sqrt{a^2 + b^2}\, u + c)\, du$.

例. $I = \int_{-1}^1 dx \int_0^{\sqrt{1 - x^2}} dy \int_1^{1 + \sqrt{1 - x^2 - y^2}} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dz$.

积分区域分析. $x \in [-1, 1]$, $0 \le y \le \sqrt{1 - x^2}$. $x^2 + y^2 \le 1$

$1 \le z \le 1 + \sqrt{1 - x^2 - y^2}$

$\Rightarrow x^2 + y^2 + (z - 1)^2 \le 1$, $y \ge 0$, $z \ge 1$. 去球.

换元 $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta. \end{cases}$　　$\frac{1}{\cos\theta} \le r \le 2\cos\theta$.
$\cos\theta \ge \frac{1}{2}$, $\cos 2\theta \ge 0$, $\theta \in [0, \frac{\pi}{4}]$.
$0 \le \varphi \le \pi$.

$= \int_0^\pi d\varphi \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r} \times r^2 \sin\theta\, dr$

$= \pi \int_0^{\frac{\pi}{4}} d\theta\, \frac{1}{2} (4\cos^2\theta - \frac{1}{\cos^2\theta}) \sin\theta$

$= -\frac{\pi}{2} \int_0^{\frac{\pi}{4}} 4\cos^2\theta - \frac{1}{\cos^2\theta}\, d\cos\theta$

$= \frac{\pi}{2} (-\frac{1}{\cos\theta} \Big|_0^{\frac{\pi}{4}} - \frac{4}{3}\cos^3\theta \Big|_0^{\frac{\pi}{4}}) = \frac{\pi}{6}(7 - 4\sqrt{2})$