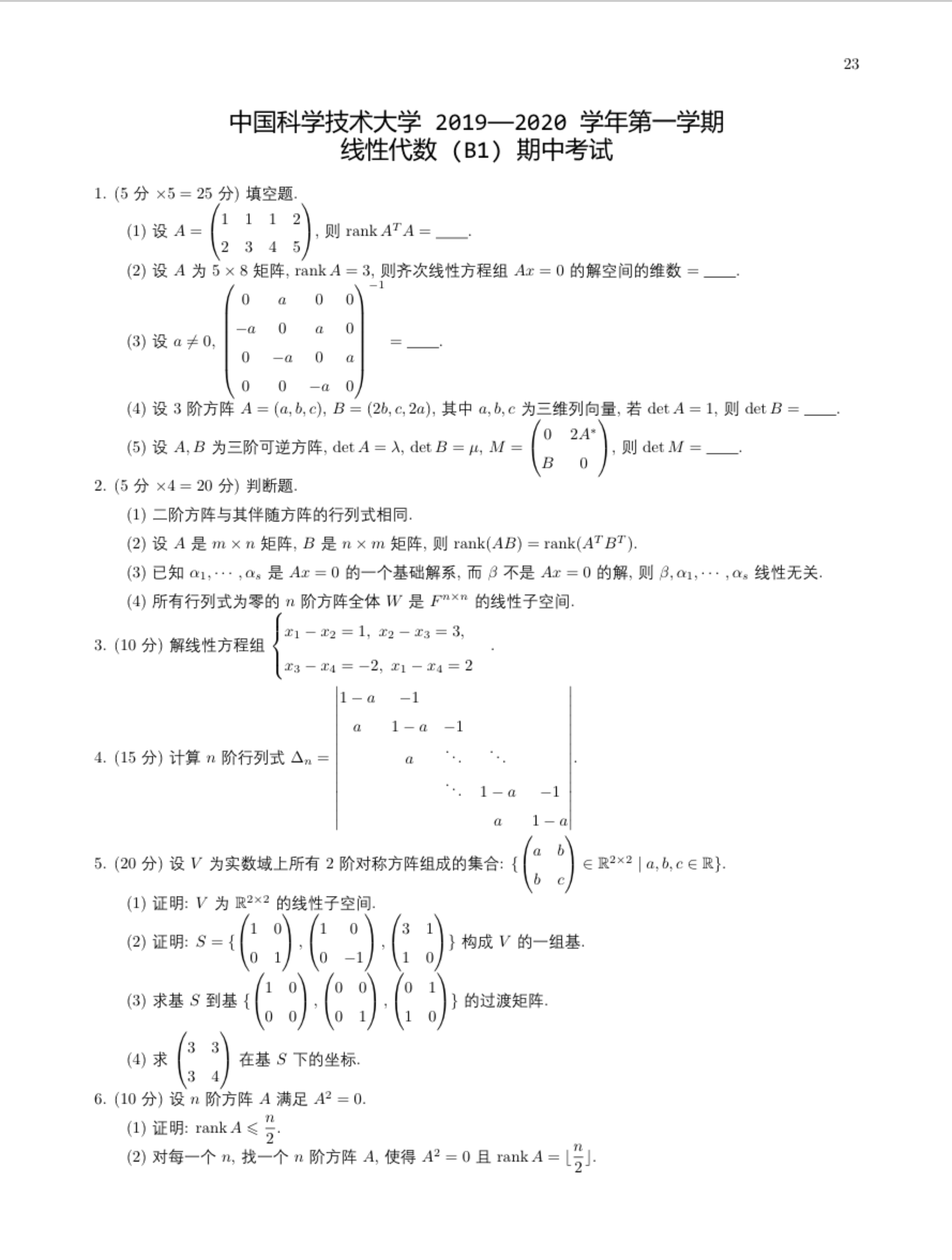
19～20期中

2022年5月11日 星期三　　下午1:33



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中国科学技术大学 2019—2020 学年第一学期
线性代数（B1）期中考试

1. (5 分 ×5 = 25 分) 填空题.

(1) 设 $A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 5 \end{pmatrix}$, 则 $\operatorname{rank} A^T A =$ ____.

(2) 设 A 为 5×8 矩阵, $\operatorname{rank} A = 3$, 则齐次线性方程组 $Ax = 0$ 的解空间的维数 = ____.

(3) 设 $a \neq 0$, $\begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & a & 0 \\ 0 & -a & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}^{-1} =$ ____.

(4) 设 3 阶方阵 $A = (a, b, c)$, $B = (2b, c, 2a)$, 其中 a, b, c 为三维列向量, 若 $\det A = 1$, 则 $\det B =$ ____.

(5) 设 A, B 为三阶可逆方阵, $\det A = \lambda$, $\det B = \mu$, $M = \begin{pmatrix} 0 & 2A^* \\ B & 0 \end{pmatrix}$, 则 $\det M =$ ____.

2. (5 分 ×4 = 20 分) 判断题.

(1) 二阶方阵与其伴随方阵的行列式相同.

(2) 设 A 是 $m \times n$ 矩阵, B 是 $n \times m$ 矩阵, 则 $\operatorname{rank}(AB) = \operatorname{rank}(A^T B^T)$.

(3) 已知 $\alpha_1, \cdots, \alpha_s$ 是 $Ax = 0$ 的一个基础解系, 而 β 不是 $Ax = 0$ 的解, 则 $\beta, \alpha_1, \cdots, \alpha_s$ 线性无关.

(4) 所有行列式为零的 n 阶方阵全体 W 是 $F^{n\times n}$ 的线性子空间.

3. (10 分) 解线性方程组 $\begin{cases} x_1 - x_2 = 1,\ x_2 - x_3 = 3, \\ x_3 - x_4 = -2,\ x_1 - x_4 = 2 \end{cases}$.

4. (15 分) 计算 n 阶行列式 $\Delta_n = \begin{vmatrix} 1-a & -1 & & & \\ a & 1-a & -1 & & \\ & a & \ddots & \ddots & \\ & & \ddots & 1-a & -1 \\ & & & a & 1-a \end{vmatrix}$.

5. (20 分) 设 V 为实数域上所有 2 阶对称方阵组成的集合: $\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a, b, c \in \mathbb{R}\}$.

(1) 证明: V 为 $\mathbb{R}^{2\times 2}$ 的线性子空间.

(2) 证明: $S = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\}$ 构成 V 的一组基.

(3) 求基 S 到基 $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ 的过渡矩阵.

(4) 求 $\begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix}$ 在基 S 下的坐标.

6. (10 分) 设 n 阶方阵 A 满足 $A^2 = 0$.

(1) 证明: $\operatorname{rank} A \leqslant \frac{n}{2}$.

(2) 对每一个 n, 找一个 n 阶方阵 A, 使得 $A^2 = 0$ 且 $\operatorname{rank} A = \lfloor \frac{n}{2} \rfloor$.

1.(1) 2.

$$\operatorname{rank}\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

$$=\operatorname{rank}\begin{pmatrix} 5 & 7 & 9 & 12 \\ 7 & 10 & 13 & 17 \\ 9 & 13 & 17 & 22 \\ 12 & 17 & 22 & 29 \end{pmatrix}$$

$\det A^T A = 0$

$m > n$.

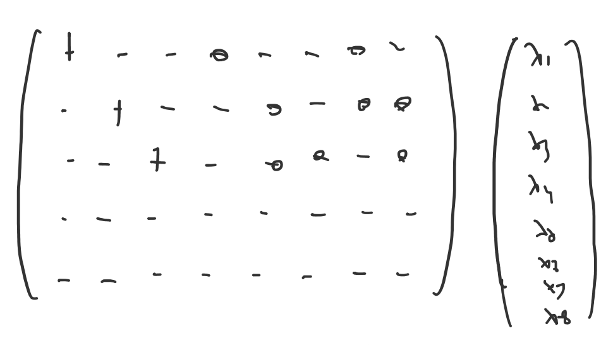
$\operatorname{rk} \leq 3$.

$$=\operatorname{rank}\begin{pmatrix} 5 & 7 & 9 & 12 \\ 0 & 10-7\times\frac{7}{5} & 13-9\times\frac{7}{5} & 17-12\times\frac{7}{5} \\ 0 & 13-7\times\frac{9}{5} & 17-7\times\frac{9}{5} & 22-12\times\frac{9}{5} \\ 0 & 17-7\times\frac{12}{5} & 22-7\times\frac{12}{5} & 29-12\times\frac{12}{5} \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 & 12 \end{pmatrix}$$

$$\begin{vmatrix} 5 & 7 & 9 \\ 7 & 10 & 13 \\ 9 & 13 & 17 \end{vmatrix} = 50\times17 + 2\times7\times9\times13 - 810 - 49\times17 - 169\times5$$
$$= 17 + 14\times13\times9 - 5\times13\times13 - 810$$
$$= 17 + 61\times13 - 810 = 0$$

$$\begin{vmatrix} 5 & 7 \\ 7 & 10 \end{vmatrix} = 1.$$

(2) 5.



(3) $$\begin{pmatrix} 0 & -\frac{1}{a} & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & \frac{1}{a} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ -a & 0 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & -a & 0 & a & 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} -a & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a & 1 & 0 & 1 & 0 \end{pmatrix}$$

(4) 4.

$$A = \begin{pmatrix} a & b & c \end{pmatrix} \qquad B = \begin{pmatrix} 2b & c & 2a \end{pmatrix} = 4\begin{pmatrix} b & c & a \\ a & b & c \end{pmatrix}$$

(5) $-2\lambda^{n-1}\mu$.

$$\det\begin{pmatrix} 0 & 2A^* \\ B & 0 \end{pmatrix} = -\begin{vmatrix} 2A^* & 0 \\ 0 & B \end{vmatrix} = -2|A^*||B|.$$

$|A||A^*| = |A|^n$. $|A^*| = |A|^{n-1} = \lambda^{n-1}$

2.(1) 正确.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| = ad - bc$.

$A^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, $|A^*| = ad - bc$

(2) 正确.

$$\begin{array}{c|cc} & u & n-u \\ u & I_u & 0 \\ m-u & 0 & 0 \end{array} \quad \begin{array}{c|cc} & v & m-v \\ v & I_v & 0 \\ n-v & 0 & 0 \end{array} = \begin{pmatrix} I_{\min(u,v)} & 0 \\ 0 & 0 \end{pmatrix}_{m\times m}.$$

P_1AQ_1, $P_2BQ_2 = I_u$; I_v.

$Q_1^T A^T P_1^T\ Q_2^T B^T P_2^T = I_u \times I_v$

(3) 正确.

A的解空间 $N(A) = \langle \vec{\alpha}_1, \vec{\alpha}_2 \ldots, \vec{\alpha}_s \rangle$　$\operatorname{rk} = s$.

β不是$Ax=0$的解, $\beta \neq 0$ 且 β不在$N(A)$中

∴ β不能用$\alpha_1 \ldots \alpha_s$表示.

同时$\alpha_1 \ldots \alpha_s$不能互相表示. ∴无关.

(4). 错误

行列式为0的n阶方阵 ⊃ 所有不可逆n阶方阵.

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$ 可逆.

3. $$\begin{cases} x_1 - x_2 = 1 \\ x_2 - x_3 = 3 \\ x_3 - x_4 = -2 \\ x_1 - x_4 = 2 \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & -1 & 2 \end{array}\right)$$

$$\to \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 & 1 & -1 & -2 \end{matrix}$$

$x_4 = t$. $x_3 = -2 + t$, $x_2 = 3 + x_3 = 3 - 2 + t$, $x_1 = 1 + x_2 = 2 + t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 1 \\ -2 \\ 0 \end{pmatrix}.$$

4. $$\Delta_n = \begin{vmatrix} 1-a & -1 & & & \\ a & 1-a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1-a & -1 \\ & & & a & 1-a \end{vmatrix}$$ $a \neq 0$ 且 $a \neq 1$.

$$= (1-a)\Delta_{n-1} + (-1)a\begin{vmatrix} -1 & & & \\ a & 1-a & -1 & \\ & \ddots & \ddots & \ddots \\ & & 1-a & -1 \\ & & a & 1-a \end{vmatrix}$$

$= (1-a)\Delta_{n-1} + a\Delta_{n-2}$.

$\Delta_n + \mu\Delta_{n-1} = (1-a+\mu)\Delta_{n-1} + \mu(1-a+\mu)\Delta_{n-2}$.

$\mu - a\mu + \mu^2 = a$, $a = \frac{\mu(1+\mu)}{1+\mu}$

$\mu = -1$, $\Delta_n - \Delta_{n-1} = (2-a)(\Delta_{n-1} - \Delta_{n-2})$　$n \geqslant 3$

$a \neq 2$, $\Delta_n - \Delta_{n-1} = (2-a)^{n-2}(\Delta_2 - \Delta_1)$
$= (2-a)^{n-2}((1-a)^2 + a - 1 + a)$
$= (2-a)^{n-2}a^2$.

$\Delta_4 - \Delta_3 = (2-a)a^2$,　$2a + (1-a)^2$

$\Delta_3 = (1-a)^3 + a(1-a) + a(1-a) = (1-a)(a^2+1)$

$\Delta_n = a^2\frac{(2-a)[(2-a)^{n-1}]}{1-a} + (1-a)(a^2+1)$

$a = 2$, $\Delta_n = \Delta_{n-1} = \Delta_1 = 1 - a = -1$.

但 $\Delta_2 = \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} = 1 + 2 = 3$ ✗

$\mu \neq -1$, $a = \mu$. $\Delta_n + a\Delta_{n-1} = \Delta_{n-1} + a\Delta_{n-2}$

$= \Delta_2 + a\Delta_1 = (1-a)^2 + a + a(1-a) = 1$.

$\Delta_n + \lambda = -a(\Delta_{n-1} + \lambda) = -a\Delta_{n-1} - a\lambda$.

$-a\lambda - \lambda = 1$, $\lambda = \frac{1}{a+1}$. $a \neq -1$.

∴ $\Delta_n + \frac{1}{a+1} = -a(\Delta_{n-1} + \frac{1}{a+1}) = (-a)^{n-2}((1+a)(1+a^2) + \frac{1}{a+1})$

$\Delta_n = (-a)^{n-2}((1+a)(1+a^2) + \frac{1}{a+1}) - \frac{1}{a+1}$

$a = 0$, $$\Delta_n = \begin{vmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & -1 \\ & & & 1 \end{vmatrix} = 1.$$

$a = 1$. $$\Delta_n = \begin{vmatrix} 0 & -1 & & \\ 1 & 0 & \ddots & \\ & \ddots & 0 & -1 \\ & & 1 & 0 \end{vmatrix}$$

$$= (-1)\times(-1)\begin{vmatrix} 1 & -1 & 0 & & \\ 0 & 0 & -1 & & \\ 0 & 1 & 0 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & 0 \end{vmatrix} = \Delta_{n-2}$$

$\Delta_1 = 0$.

$\Delta_2 = 1$.

$$\Delta_n = \begin{cases} 0, & n为奇 \\ 1, & n为偶. \end{cases}$$

5.(1) 对 $A \in V$: $A_{2\times2}$, $A^T = A$. $B \in V$.

$(A+B)^T = A^T + B^T \in V$,

$\lambda A^T = (\lambda A)^T = \lambda A \in V$. 构成子空间.

(2) 对方程 $\lambda_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_3\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{cases} \lambda_1 + \lambda_2 + 3\lambda_3 = 0 \\ \lambda_3 = 0 \\ \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow$$ $\lambda_1 = \lambda_2 = \lambda_3 = 0$. 线性无关的矩阵.

且 $\dim V = 2 + 1 = 3$. $\operatorname{rk} S = 3$.

∴ S为V的一组基.

(3) $$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\}$$

$$= \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

过渡矩阵T满足

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\}T = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$$

$$T = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$

(4) $$\begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix} = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x_3\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + 3x_3 = 3 \\ x_3 = 3 \\ x_1 - x_2 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = -1 \\ x_2 = -5. \\ x_3 = 3 \end{cases}$$

∴坐标 $(-1, -5, 3)^T$.

6.(1) $A \cdot A = 0$. A的各列向量为$Ax=0$的解.

记为解空间 $N(A)$.　　$\operatorname{rk} N(A) \geqslant \frac{n}{2}$.

则 $\operatorname{rank}(N(A)) + \operatorname{rank} A = n$.

则对 $\operatorname{rank}(N(A))$, $A(\alpha_1\ \alpha_2 \ldots \alpha_n) = 0$.

$\alpha_1 \sim \alpha_n$中至少有几个向量作为基础解系,

才能确定所有的$\alpha_1 \ldots \alpha_n$.

∴ $\operatorname{rank}(N(A)) \geqslant \frac{n}{2}$, $\operatorname{rk} A \leqslant \frac{n}{2}$.

(2) $$A = \begin{pmatrix} I_{\frac{n}{2}} & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & -I_{\frac{n}{2}} \end{pmatrix}$$