18～19春期中

2022年5月10日 星期二　　下午4:16

中国科学技术大学 2018—2019 学年第二学期 线性代数 (B1) 期中考试

1. (5 分 ×5 = 25 分) 填空题.

(1) 方阵的方幂 $\begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix}^{2019} =$ ____.

(2) 已知矩阵 $A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ -2 & 1 & -1 & 1 \\ 1 & -2 & 2 & -1 \\ 0 & 0 & 3 & 4 \end{pmatrix}$, A_{ij} 表示 (i,j) 元的代数余子式, 则 $A_{11} - A_{12} =$ ____.

(3) 若方阵 $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$, 则 $A^{-1} =$ ____.

(4) 设 $A \in \mathbb{C}^{4\times 4}$, $\operatorname{rank} A = 3$, 则 $A^* x = 0$ 的解空间维数是 ____.

(5) 若向量组的秩 $\operatorname{rank}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 4$, 则 $\operatorname{rank}(\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_4+\alpha_1) =$ ____.

2. (5 分 ×4 = 20 分) 判断题.

(1) 设 A, B 均为行满秩的 $m \times n$ 实矩阵, 则 $\det(AB^T) \neq 0$.

(2) 设非零矩阵 A, B 满足 $AB = 0$, 则 A 的列向量组线性相关.

(3) 设 $A \in \mathbb{R}^{5\times 3}$, 若 $Ax = 0$ 有非零解, 则对所有 $b \in \mathbb{R}^5$, $Ax = b$ 存在无穷多解.

(4) 对于任意 n 阶实方阵 A, B, 不存在非零的实数 μ, 使得 $AB - BA = \mu I_n$.

3. (15 分) 考虑 $\begin{cases} \lambda x_1 + x_2 + x_3 = \lambda - 3 \\ x_1 + \lambda x_2 + x_3 = -2 \\ x_1 + x_2 + \lambda x_3 = -2 \end{cases}$, 问 λ 为何值时, 该方程组有唯一解? 无解? 有无穷多解? 在有解的情况下, 给出通解.

4. (15 分) 计算行列式 $\begin{vmatrix} x_1 - 1 & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 - 2 & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3 - 3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n - n \end{vmatrix}$.

5. (15 分) 设 $F[x]_4$ 为数域 F 上次数小于 4 的多项式在加法和数乘下构成的线性空间.

(1) 证明 $S = \{1, 1+x, (1+x)^2, (1+x)^3\}$ 构成了该线性空间的一组基.

(2) 求基 $1, x, x^2, x^3$ 到基 S 的过渡矩阵 T.

(3) 求多项式 $1 + x + x^2 + x^3$ 在基 S 下的坐标.

6. (10 分) 设 A 是 n 阶方阵, I_n 为 n 阶单位矩阵.

(1) 若 $A^2 = A$, 证明: $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$.

(2) 反之, 若 $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$, 证明: $A^2 = A$.

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1. (1) ... (2) ... (3) ...

[illegible handwritten computations for 1(1)–(3)]

$\to \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & -1 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & -1 & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -\frac{9}{7} & \frac{3}{7} & \frac{6}{7} \\ 0 & 0 & 7 & -1 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 7 & 0 & -3 & 1 & 2 \\ 0 & 0 & 7 & -1 & -2 & 3 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \\ 0 & 7 & 0 & -3 & 1 & 2 \\ 0 & 0 & 7 & -1 & -2 & 3 \end{pmatrix}$　　$-3 \times \frac{2}{7} = -\frac{6}{7}$

(4) 3.

$\operatorname{rk} A = 3$, $\operatorname{rk} A^* = 1$.

(5) 4.

$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$ 无关.

2. (1) 错误

$A_{m\times n}$ 行满秩. $A = (I_m\ R_{m\times(n-m)})$　$B = (I_m\ S_{m\times(n-m)})$

$A \times B^T = (I_m\ R)\begin{pmatrix} I_m & 0 \\ S & 0 \end{pmatrix} = \begin{pmatrix} I_m + RS & 0 \\ 0 & 0 \end{pmatrix}$　$\det = 0$

(2) 正确.

$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}(\beta_1 \cdots \beta_n) = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_n \\ \alpha_2\beta_1 & & \cdots & \alpha_n\beta_n \end{pmatrix}$

$AB = 0 \to B$ 各列向量都为 A 的解向量.

$B \neq 0$, A 的列空间有非零向量. 线性相关.

(3) 正确.

$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$　各行向量线性相关

列向量组线性组合 $\to b$.

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$,　$x_1 = -x_3$.　$x_2 = 0$

对 b 言, 列向量线性组合无多余组.

(4) 正确.

反之 AB.　$AB - BA = \mu I_n \to \mu = 0$.

假设 $\exists A, B$, 当 $AB - BA = \mu I_n$ 时, $\mu \neq 0$.

$\sum_{k=1}^n a_{ik}b_{kj} - \sum_{k=1}^n b_{ik}a_{kj} = \delta_{ij}\mu$.

$i = j$ 时, $\sum_{k=1}^n a_{ik}b_{ki} - \sum_{k=1}^n a_{ki}b_{ik} = \mu$.

且 $i \neq j$ 时, $\sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^n b_{ik}a_{kj}$

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$\therefore A$ 或 B 必有一个行/列向量线性相关.

$\therefore AB = BA$. $\mu = 0$.

3. $\left(\begin{array}{ccc|c} \lambda & 1 & 1 & \lambda-3 \\ 1 & \lambda & 1 & -2 \\ 1 & 1 & \lambda & -2 \end{array}\right) \to \begin{pmatrix} 1 & 1 & \lambda & -2 \\ 1 & \lambda & 1 & -2 \\ \lambda & 1 & 1 & \lambda-3 \end{pmatrix}$

$\lambda = 0$,　$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

$x_3 = -1$,　$x_2 = x_3 = -1$,　$x_1 = -2 - x_2 = -1$. 唯一.

$\lambda \neq 0$,　$\begin{pmatrix} 1 & 1 & \lambda & -2 \\ 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda^2 & 3(\lambda-1) \end{pmatrix}$

$\lambda = 1$,　$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$　无穷.

$\lambda \neq 1$,　$\begin{pmatrix} 1 & 1 & \lambda & -2 \\ 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 0 & -(\lambda+2)(\lambda-1) & 3(\lambda-1) \end{pmatrix}$

$\lambda = -2$ 时, 方程组无解.

$\lambda \neq -2$ 时,　$x_3 = -\frac{3}{\lambda+2}$　唯一.

$x_2 = \frac{3}{\lambda+2}$

$x_1 = -2 + \frac{3\lambda}{\lambda+2} - \frac{3}{\lambda+2} = \frac{\lambda-5}{\lambda+2}$.

4. $\begin{vmatrix} x_1-1 & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2-2 & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3-3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n-n \end{vmatrix}$

$= \begin{vmatrix} x_1-1 & x_2 & x_3 & \cdots & x_n \\ 1 & -2 & 0 & \cdots & 0 \\ 0 & 2 & -3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & n-1 & -n \end{vmatrix}$

$-2(-1)^{n-2} + (-1)^{n-3}$

$(-1)^{k-1}(-1)^{k-1}$

Δ_2

$= (x_1-1)(-1)^{n-1}\prod_{i=2}^n i - 1 \times \begin{vmatrix} x_2 & x_3 & \cdots & x_n \\ 2 & -3 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & n-1 & -n \end{vmatrix}$

$= (x_1-1)(-1)^{n-1}\prod_{i=2}^n i - x_2(-1)^{n-2}\prod_{i=3}^n i + 2 \times \Delta_3$

$= (x_1-1)(-1)^{n-1}\prod_{i=2}^n i + \sum_{j=2}(j-1)(-1)^{j-1}x_j(-1)^{n-j}\prod_{i=j+1}^n i$

$= (x_1-1)(-1)^{n-1}\prod_{j=2}^n i + \sum_{j=2}^n (j-1)(-1)^{n-1}x_j\prod_{i=j+1}^n i$.

5. (1) 对 $\lambda_1 1 + \lambda_2(1+x) + \lambda_3(1+x)^2 + \lambda_4(1+x)^3 = 0$,

$(\lambda_1+\lambda_2+\lambda_3+\lambda_4) + (\lambda_2+2\lambda_3+3\lambda_4)x + (\lambda_3+3\lambda_4)x^2 + \lambda_4 x^3 = 0$.

则 $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. 反方向可. 二无关.

次数不超过 3, $\dim F_4[x] = 4$.

$\therefore S$ 中的向量为极大无关组, 构成一组基.

(2) $(1, 1+x, (1+x)^2, (1+x)^3)$

$= (1, x, x^2, x^3)\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$T^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,　$T = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(3) $1 + x + x^2 + x^3 = (1, x, x^2, x^3)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$= (1, (1+x), (1+x)^2, (1+x)^3)\, T\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ 坐标为 $T\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 1 \end{pmatrix}$

6. (1) $\operatorname{rk} A + \operatorname{rk}(I_n - A) = \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & I_n - A \end{pmatrix} = \operatorname{rk}\begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} = n$.

$\to \begin{pmatrix} A & 0 \\ A & I_n - A \end{pmatrix} \to \begin{pmatrix} A & A \\ A & I_n \end{pmatrix} \to \begin{pmatrix} A - A^2 & 0 \\ A & I_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A & I_n \end{pmatrix}$

(2) $\operatorname{rk} A = n$ 时,　$\operatorname{rk}(I_n - A) = 0$.

$\therefore A = I_n$,　$A^2 = A$.

$\operatorname{rk}(I_n - A) = n$ 时,　$\operatorname{rk} A = 0$.

$\therefore I_n - A = I_n$,　$A^2 = A$.

$1 \le \operatorname{rk} A \le n-1$ 时,　$\operatorname{rk}\begin{pmatrix} A & 0 \\ 0 & I_n - A \end{pmatrix} = \operatorname{rk}\begin{pmatrix} A - A^2 & 0 \\ 0 & I_n \end{pmatrix} = n$.

$A - A^2 = 0$,　$A = A^2$.