

中国科学技术大学 2016—2017 学年第二学期
线性代数 (B1) 期中考试

1. (4分 × 6 = 24分) 填空题
- (1) $\alpha_1 = (1, 3, 2)^T, \alpha_2 = (4, 4, 0)^T, \alpha_3 = (2, 5, 3)^T, \alpha_4 = (-1, 2, 3)^T$, 则 $\text{rank}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad}$.
- (2) $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix}$, 则 $A^{10} = \underline{\quad}$.
- (3) 设 A 为 n 阶方阵, $\det A = 5$, A^* 为 A 的伴随方阵, 则 $\det A^* = \underline{\quad}$.
- (4) 设 $A = \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & 2 \\ 1 & 5 & 2 & 3 \end{pmatrix}$, A_{ij} 为代数余子式, 则 $A_{11} - 3A_{21} + 2A_{31} - A_{41} = \underline{\quad}$.
- (5) 若向量 $\beta = (3, 9, 6)$ 不能由向量组 $\alpha_1 = (1, 1, 2), \alpha_2 = (1, 2, -1), \alpha_3 = (1, -\lambda, 3)$ 线性表示, 则 $\lambda = \underline{\quad}$.
- (6) 设分块矩阵 $A = \begin{pmatrix} O & B \\ C & O \end{pmatrix}$, 其中 B, C 为 n 阶可逆方阵, O 为零方阵, 则 $(A^T)^{-1} = \underline{\quad}$.
2. (5分 × 4 = 20分) 判断题 (判断下列命题是否正确, 并简要给出理由).
- (1) 设 $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 5 & -2 \\ 5 & 11 & -4 \end{pmatrix}, B = \begin{pmatrix} 5 & 0 & 4 \\ 3 & 0 & 2 \end{pmatrix}$, 则 A 与 B 不相抵.
- (2) 设数域 F 中的向量组 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性相关, $A \in F^{m \times 1}, (\beta_1, \beta_2, \dots, \beta_l) = (\alpha_1, \alpha_2, \dots, \alpha_m)A$, 则向量组 $\beta_1, \beta_2, \dots, \beta_l$ 也线性相关.
- (3) A, B 为 n 阶方阵, 则 $\text{rank}(AB) = \text{rank}(BA)$.
- (4) 设向量组 $\alpha_1, \alpha_2, \dots, \alpha_r$ 的秩为 r , 且任何向量 $\alpha_i (1 \leq i \leq r)$ 均可以被 $\alpha_1, \alpha_2, \dots, \alpha_r$ 线性表示, 则 $\alpha_1, \alpha_2, \dots, \alpha_r$ 是 $\alpha_1, \alpha_2, \dots, \alpha_r$ 的一个极大线性无关组.
3. (12分) 当 α 取何值时, $\begin{cases} 3x_1 + 5x_2 - 2x_3 - 2x_4 = 0 \\ 2x_1 + 8x_2 - 2x_3 + x_4 = \alpha \end{cases}$ 有解? 求出它的通解.
4. (16分) 设 n 阶方阵 $A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & -1 \\ -1 & -1 & -1 & \dots & -1 & -1 \end{pmatrix}$, 求 $\det A$ 及 A^{-1} .
5. (16分) 设 $P_n[x]$ 为实数域 \mathbb{R} 上次数不超过 n 的多项式全体, 按多项式的加法数域构成线性空间.
- (1) 证明: $S = \{1, x+1, (x+1)^2, (x+1)^3\}$ 构成 $P_3[x]$ 上的一组基;
- (2) 求基 S 到自然基 $\{1, x, x^2, x^3\}$ 的过渡矩阵 T ;
- (3) 求多项式 $5+7x-x^2+13x^3$ 在基 S 下的坐标.
6. 设方阵 $A = (a_{ij})_{n \times n}, c = \text{tr}(A) = \sum_{i=1}^n a_{ii}$, 已知 $\text{rank} A = 1$,

- (1) 证明: $A^2 = cA$;
- (2) 计算 $\det(I+A)$, 其中 I 为 n 阶单位方阵.

1. (1) 2

$$\text{rk} \begin{pmatrix} 1 & 4 & 2 & -1 \\ 3 & 4 & 5 & 2 \\ 2 & 0 & 3 & 3 \end{pmatrix}$$

$$= \text{rk} \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & -8 & -1 & 5 \\ 0 & -8 & -1 & 5 \end{pmatrix}$$

(2) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix} \begin{matrix} \alpha \\ \beta \\ \beta \end{matrix}$$

$$A^2 = \alpha \beta \alpha \beta - \alpha \beta$$

$$\beta \alpha = (1 \ 2 \ 1) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 1 - 2 + 2 = 1$$

(3) 5^{n-1}

$|A| = 5, |A^2| = |A|^2 = 5^2, |A^3| = |A|^3 = 5^3$

(4) 6

$$\begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & \frac{5}{2} \\ 1 & 5 & 2 & 7 \end{pmatrix} \xrightarrow{r_4 - r_1} \begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_2 \times (-1)} \begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & -3 & -1 & 4 \\ 0 & 0 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 \times (-1)} \begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_2 \times (-1)} \begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & -3 & -1 & 4 \\ 0 & 0 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 \times (-1)} \begin{pmatrix} 1 & 5 & 2 & 7 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(5) $-\frac{1}{3}$

$$\mu_1(1, 1, 2) \rightarrow \mu_2(1, 2, -1) \rightarrow \mu_3(1, -1, 3) = (3, 9, 6)$$

$$\begin{cases} \mu_1 + \mu_2 - \mu_3 = 3 \\ \mu_1 + 2\mu_2 - \mu_3 = 9 \\ 2\mu_1 - \mu_2 + 3\mu_3 = 6 \end{cases} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 9 \\ 2 & -1 & 3 & 6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 6 \\ 0 & -3 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 6 \\ 0 & 1 & -1 & 6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & -1 + \frac{1}{3} & 7 \end{pmatrix} \xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & -1 + \frac{1}{3} & 7 \end{pmatrix} \xrightarrow{\times (-\frac{1}{3})} \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -1 \\ 0 & 0 & -1 + \frac{1}{3} & 7 \end{pmatrix}$$

(6) $\begin{pmatrix} 0 & B^{-1} \\ C^{-1} & 0 \end{pmatrix}$

$$(A^T)^{-1} = \begin{pmatrix} 0 & C^{-1} \\ B & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & C^{-1} & 1 & 0 \\ B & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & C^{-1} & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & C^{-1} & 0 & 0 \\ 0 & C^{-1} & 2C^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & B^{-1} \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

2. (1) 相似

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 5 & -2 \\ 5 & 11 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5 & -2 \\ 2 & 3 & -1 \\ 5 & 11 & -4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 5 & -2 \\ 0 & -7 & 5 \\ 0 & -4 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5 & -2 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \end{pmatrix} \text{rk } A = 2$$

$$B = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 14 & 0 \\ 0 & 8 & 0 \end{pmatrix} \text{rk } B = 2$$

(2) 正交

$$\beta_1 \cdot \beta_2 = 0 \rightarrow \beta_1 \perp \beta_2 = \vec{b}$$

$$\begin{cases} \beta_1 = a_{11}\vec{e}_1 + \dots + a_{1m}\vec{e}_m = \sum_{i=1}^m a_{1i}\vec{e}_i \\ \vdots \\ \beta_2 = a_{21}\vec{e}_1 + \dots + a_{2m}\vec{e}_m \end{cases}$$

$$\mu_1 \beta_1 \rightarrow \mu_2 \beta_2 = \sum_{j=1}^m \mu_j \beta_j$$

$$= \sum_{j=1}^m \mu_j \sum_{i=1}^m a_{ij} \vec{e}_i$$

$$= \sum_{i=1}^m \sum_{j=1}^m \mu_j a_{ij} \vec{e}_i \Rightarrow \text{两方程}$$

(3) 相似

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{rank } AB = \text{rk} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$$

$$\text{rk } BA = \text{rk} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \text{rk} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

(4) 相似

假设相似, $\alpha_1 \sim \alpha_2 \rightarrow \alpha_1 \dots \alpha_n (m \times r)$

$m \times \alpha_1 \dots \alpha_n$ 与 $\alpha_1 \dots \alpha_n$ 的秩为 $m \times n$

$\therefore \alpha_1 \dots \alpha_n$ 为无关, 矛盾

3. $\begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 3 & 8 & -1 & 2 & 0 \\ 2 & 5 & -2 & 1 & a \end{pmatrix}$

$$\rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & 2 & 8 & -10 & -b \\ 0 & 1 & 4 & -7 & a-4 \end{pmatrix} \begin{matrix} -5a-7 \\ -3a-3 \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & 1 & 4 & -7 & -3 \\ 0 & 0 & 0 & 0 & a-\frac{1}{5} \end{pmatrix} \begin{matrix} -4-4\frac{a}{5} \\ -4-4\frac{a}{5} \end{matrix}$$

特征 $\Rightarrow a = -\frac{1}{5}$

$$\lambda_1 = 4t_1, \lambda_2 = 4t_2, \lambda_3 = -3+5t_1-4t_2$$

$$\lambda_1 = 2-4t_1+t_2, \lambda_2 = 2(-3+5t_1-4t_2)$$

$$= 2-4t_1+t_2, \lambda_3 = b-10t_1+8t_2$$

$$\vec{b} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -14 \\ 5 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} 8 \\ -3 \\ 0 \end{pmatrix}$$

4. $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \dots & -1 & -1 \\ 1 & -1 & -1 & \dots & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{vmatrix}$

$$= |(-1)^{m+1} \begin{vmatrix} -2 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -2 \end{vmatrix}| = 2^{m+1} |(-1)^{m+1} (-2)^{m+1}| = 2^{2m+2}$$

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -2 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \\ \vdots \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 \\ 0 & 0 & 0 & \dots & 1 & \frac{1}{2} \end{pmatrix} \begin{matrix} \\ \\ \\ \frac{1}{2} \\ \frac{1}{2} \end{matrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{1}{2} \end{pmatrix}$$

5. (1) $\text{rk } V = 4$. S 中 4 个 \vec{e}_i

$$\beta_1 + \beta_2(\lambda+1) + \beta_3(\lambda+1)^2 + \beta_4(\lambda+1)^3 = \vec{0}$$

取 $\beta_1 = \lambda, \beta_2 = \beta_3 = \beta_4 = 1$

(2) $U = (\lambda+1)^2, (\lambda+1)^3$

$$= (1, \lambda, \lambda^2, \lambda^3) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (1, \lambda, \lambda^2, \lambda^3)^T$$

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) $\vec{b} + \lambda x + \lambda^2 x^2 + \lambda^3 x^3$

$$= (1, \lambda, \lambda^2, \lambda^3) \begin{pmatrix} 0 \\ 7 \\ -1 \\ 13 \end{pmatrix} \begin{matrix} 5-7-1-13 \\ 7+2+19 \\ -1-19 \\ 11 \end{matrix}$$

$$= (1, \lambda, \lambda^2, \lambda^3)^T \begin{pmatrix} 0 \\ 7 \\ -1 \\ 13 \end{pmatrix}$$

$$= (1, \lambda, \lambda^2, \lambda^3)^T \begin{pmatrix} -16 \\ 48 \\ -40 \\ -13 \end{pmatrix}$$

特征 $(-16, 48, -40, -13)$

6. $\text{tr}(A) = \sum_{i=1}^n a_{ii}$, $\text{rate } A = 1$

(1) $\text{rank } A = 1, A = \alpha\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1, \dots, b_n)$

$$a_{ii} = a_i b_i, \sum_{i=1}^n a_i b_i = c$$

$$A^2 = \alpha\beta\alpha\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1, \dots, b_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1, \dots, b_n)$$

$$= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \sum_{i=1}^n b_i a_i (b_1, \dots, b_n)$$

$$= c \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (b_1, \dots, b_n) = cA$$

(2) $\det(I + \beta^T \alpha) = \det(I + \alpha\beta)$

$$= \det(1 + \beta^T \alpha^T) = 1 + c$$