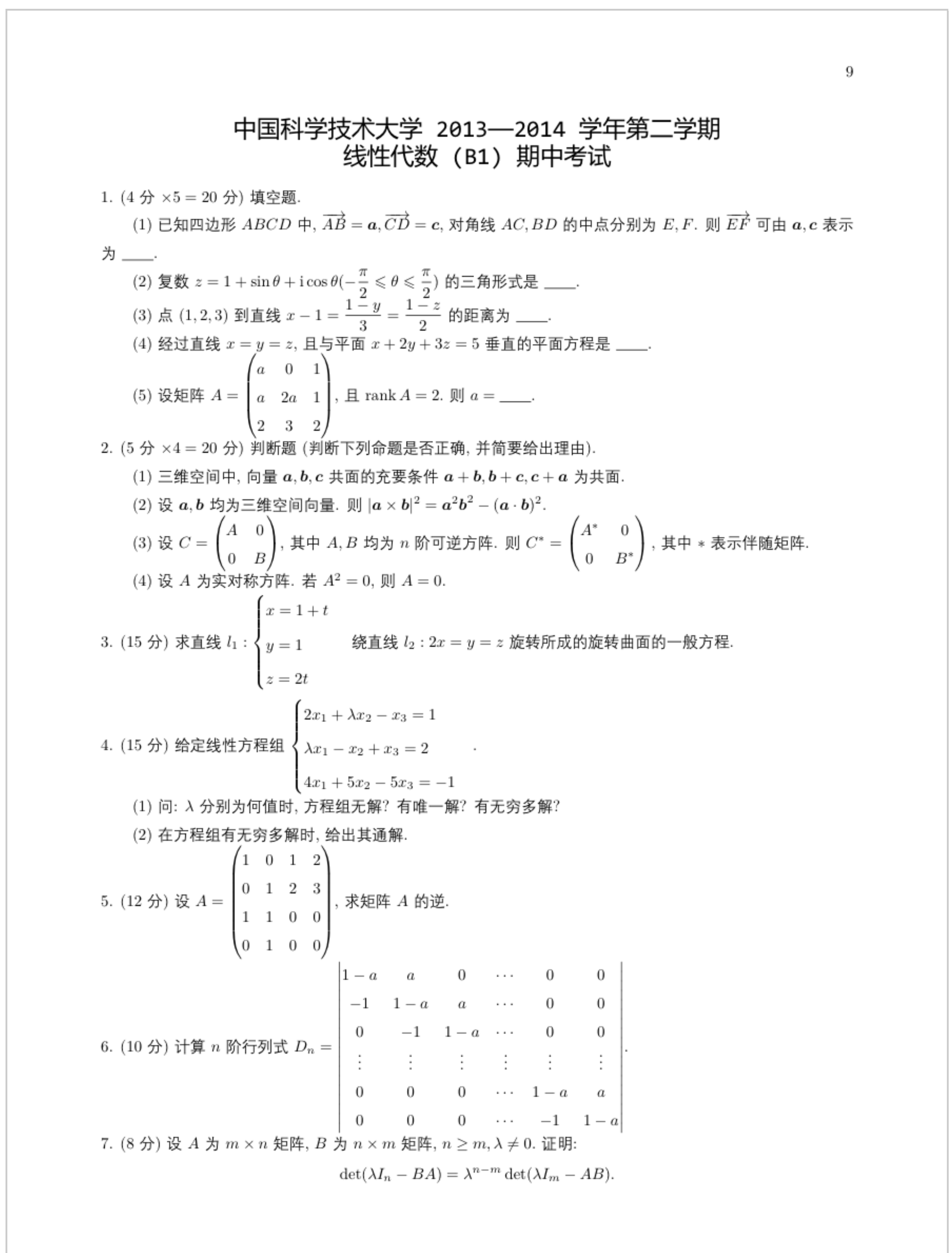
14～15期中

2022年5月2日 星期一　　下午6:56



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中国科学技术大学 2013—2014 学年第二学期
线性代数（B1）期中考试

1. (4 分 ×5 = 20 分) 填空题.

(1) 已知四边形 $ABCD$ 中, $\overrightarrow{AB} = \boldsymbol{a}, \overrightarrow{CD} = \boldsymbol{c}$, 对角线 AC, BD 的中点分别为 E, F. 则 $\overrightarrow{EF}$ 可由 $\boldsymbol{a}, \boldsymbol{c}$ 表示为 ____.

(2) 复数 $z = 1 + \sin\theta + \mathrm{i}\cos\theta(-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2})$ 的三角形式是 ____.

(3) 点 $(1, 2, 3)$ 到直线 $x - 1 = \frac{1-y}{3} = \frac{1-z}{2}$ 的距离为 ____.

(4) 经过直线 $x = y = z$, 且与平面 $x + 2y + 3z = 5$ 垂直的平面方程是 ____.

(5) 设矩阵 $A = \begin{pmatrix} a & 0 & 1 \\ a & 2a & 1 \\ 2 & 3 & 2 \end{pmatrix}$, 且 $\operatorname{rank} A = 2$. 则 $a =$ ____.

2. (5 分 ×4 = 20 分) 判断题 (判断下列命题是否正确, 并简要给出理由).

(1) 三维空间中, 向量 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 共面的充要条件 $\boldsymbol{a} + \boldsymbol{b}, \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{c} + \boldsymbol{a}$ 为共面.

(2) 设 $\boldsymbol{a}, \boldsymbol{b}$ 均为三维空间向量. 则 $|\boldsymbol{a} \times \boldsymbol{b}|^2 = \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

(3) 设 $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, 其中 A, B 均为 n 阶可逆方阵. 则 $C^* = \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix}$, 其中 $*$ 表示伴随矩阵.

(4) 设 A 为实对称方阵. 若 $A^2 = 0$, 则 $A = 0$.

3. (15 分) 求直线 $l_1 : \begin{cases} x = 1 + t \\ y = 1 \\ z = 2t \end{cases}$ 绕直线 $l_2 : 2x = y = z$ 旋转所成的旋转曲面的一般方程.

4. (15 分) 给定线性方程组 $\begin{cases} 2x_1 + \lambda x_2 - x_3 = 1 \\ \lambda x_1 - x_2 + x_3 = 2 \\ 4x_1 + 5x_2 - 5x_3 = -1 \end{cases}$.

(1) 问: λ 分别为何值时, 方程组无解? 有唯一解? 有无穷多解?

(2) 在方程组有无穷多解时, 给出其通解.

5. (12 分) 设 $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, 求矩阵 A 的逆.

6. (10 分) 计算 n 阶行列式 $D_n = \begin{vmatrix} 1-a & a & 0 & \cdots & 0 & 0 \\ -1 & 1-a & a & \cdots & 0 & 0 \\ 0 & -1 & 1-a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-a & a \\ 0 & 0 & 0 & \cdots & -1 & 1-a \end{vmatrix}$.

7. (8 分) 设 A 为 $m \times n$ 矩阵, B 为 $n \times m$ 矩阵, $n \geq m, \lambda \neq 0$. 证明:

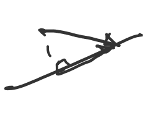
$$\det(\lambda I_n - BA) = \lambda^{n-m} \det(\lambda I_m - AB).$$

1. (1) 三维空间内平行六面体的体积.

(2) $z = \begin{cases} \pm 1, & n\text{为偶数} \\ \pm i, & n\text{为奇数} \end{cases}$.

(3) $\frac{17}{\sqrt{77}}$

$\vec{v} = (4, 5, 6)$, 过 $(-1, -2, -3)$.

$\vec{a} = (0, 1, 2)$  $d = \frac{15+121}{\sqrt{16+36+25}} = \frac{17}{\sqrt{77}}$

(4) $f(x^2+y^2, z) = 0$

(5) 双曲面.

(6) $\begin{cases} r, & r = n \\ 1, & r = n-1 \\ 0, & r \leq n-2 \end{cases}$

2. (1) 错误. 如 $\vec{a} = (1, 0)$, $\vec{b} = (0, 1)$, $\vec{c} = (1, 1)$.

$(\vec{a} \cdot \vec{b}) \cdot \vec{c} = \vec{0}$, $\vec{a} \cdot (\vec{b} \cdot \vec{c}) = (1, 0)$.

(2) 正确. $\vec{a} = (x_1, y_1, z_1)$ $\vec{b} = (x_2, y_2, z_2)$ $\vec{c} = (x_3, y_3, z_3)$

$\vec{a} \times \vec{b} = (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1)$

$\vec{b} \times \vec{c} = (y_2z_3 - y_3z_2, x_3z_2 - x_2z_3, x_2y_3 - x_3y_2)$

$\vec{a} \times \vec{c} = (y_1z_3 - y_3z_1, x_3z_1 - x_1z_3, x_1y_3 - x_3y_1)$.

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ y_1z_2 - y_2z_1 & x_2z_1 - x_1z_2 & x_1y_2 - x_2y_1 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$= (z_3(x_2z_1 - x_1z_2) - y_3(x_1y_2 - x_2y_1),$

$x_3(x_1y_2 - x_2y_1) - z_3(y_1z_2 - y_2z_1),$

$y_3(y_1z_2 - y_2z_1) - x_3(x_2z_1 - x_1z_2))$.

$\vec{a} \times (\vec{c} \times \vec{b}) - \vec{b} \times (\vec{c} \times \vec{a})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 & y_2 & z_2 \\ y_1z_3 - y_3z_1 & x_3z_1 - x_1z_3 & x_1y_3 - x_3y_1 \end{vmatrix} - \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ y_2z_3 - y_3z_2 & x_3z_2 - x_2z_3 & x_2y_3 - x_3y_2 \end{vmatrix}$$

$= (y_2(x_1y_3 - \underline{x_3y_1}) - z_2(\underline{x_3z_1} - x_1z_3) - y_1(x_2y_3 - \underline{x_3y_2}) + z_1(\underline{x_3z_2} - x_2z_3),$

$z_2(y_1z_3 - y_3z_1) - x_2(x_1y_3 - x_3y_1) - z_1(y_2z_3 - y_3z_2) + x_1(x_2y_3 - x_3y_2),$

$x_2(x_3z_1 - x_1z_3) - y_2(y_1z_3 - y_3z_1) - x_1(x_3z_2 - x_2z_3) + y_1(y_2z_3 - y_3z_2))$

$\vec{i}$ 方向分量为 $z_3(x_1z_2 - x_2z_1) - y_3(x_2y_1 - x_1y_2)$

$+ x_3(-y_1y_2 - z_1z_2 + y_1y_2 + z_1z_2)$

$= z_3(x_1z_2 - x_2z_1) - y_3(x_2y_1 - x_1y_2)$

同理, $\vec{j}$ 方向分量为 $x_3(x_1y_2 - x_2y_1) - z_3(y_1z_2 - y_2z_1)$,

$\vec{k}$ 方向分量为 $y_3(y_1z_2 - y_2z_1) - x_3(x_2z_1 - x_1z_2)$.

$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = \vec{b} \times (\vec{a} \times \vec{c}) - \vec{a} \times (\vec{b} \times \vec{c})$

$= \vec{a} \times (\vec{c} \times \vec{b}) - \vec{b} \times (\vec{c} \times \vec{a})$.

(3) 错误. $A_{ij} = a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

$B_{ij} = b_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$.

$C = AB$, $C_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = c_{ij}$

$D = BA$, $D_{ij} = \sum_{l=1}^{m} b_{il}a_{lj} = d_{ij}$.

则 $\det(AB) = \begin{vmatrix} \sum a_{1k}b_{k1} & \cdots & \sum a_{1k}b_{kn} \\ \vdots & & \vdots \\ \sum a_{nk}b_{k1} & \cdots & \sum a_{nk}b_{kn} \end{vmatrix}$

$\det(BA) = \begin{vmatrix} \sum a_{k1}b_{1k} & \cdots & \sum a_{km}b_{1k} \\ \vdots & & \vdots \\ \sum a_{k1}b_{mk} & \cdots & \sum a_{km}b_{mk} \end{vmatrix}$

$\det(AB) \neq \det(BA)$

假设 A 的第一行为0, B 的第一列为零,

$\det(AB) = 0$, $\det(BA)$ 可能不为零.

(4) 错误. $A = a_{ij}$, $B = b_{ij}$.

$C = AB$, $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$. $C^T_{ij} = \sum_{k=1}^{m} a_{jk}b_{ki}$.

$A^T = (a_{ji})_{n \times m}$, $B^T = (b_{ji})_{m \times n}$

$A^TB^T = \sum_{k=1}^{n} a_{jk}b_{ki}$

$\therefore \sum_{k=1}^{n} a_{jk}b_{ki} \neq \sum_{k=1}^{n} a_{jk}b_{ki}$.

(5) 正确.

A 可逆时, $AA^* = |A|I$, $|A^*||A| = |A|^n$, $|A^*| = |A|^{n-1}$.

A 不可逆时, $|A| = 0$.

若 $\operatorname{rank} A = n-1$, 则 $\operatorname{rank} A^* = 1$, $\det(A^*) = 0$

$\operatorname{rank} A \leq n-2$, 则 $\operatorname{rank} A^* = 0$, $\det(A^*) = 0$.

$\therefore |A^*| = |A|^{n-1} = 0$.

3. $\det(A) = 6 + 6 + 24 - 18 - 12 - 4 = 2 \neq 0$,

$\det(B) = 6 - 5 = 1$, A, B 可逆.

$AXB = C$, $XB = A^{-1}C$, $X = A^{-1}CB^{-1}$.

$$\therefore X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}^{-1}$$

$\begin{matrix} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{matrix}$

$\begin{matrix} 2 & 1 & 1 & 0 \\ 0 & 3-\frac{5}{2} & -\frac{5}{2} & 1 \end{matrix}$

$\begin{matrix} 2 & 0 & 6 & -2 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{matrix}$

$\begin{matrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{matrix}$

$$= \begin{pmatrix} -\frac{1}{3} & -2 & 2 \\ \frac{1}{6} & -\frac{4}{3} & -\frac{5}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 3 & -2 \\ 6 & 6 & -2 \\ 4 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 4 \\ 12 & 16 \\ 0 & 14 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 6 & 8 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 3 \\ -22 & 10 \\ -35 & 14 \end{pmatrix}$$

4. $\left(\begin{array}{ccc|c} a & 1 & 1 & -1 \\ 1 & a & 1 & -2 \\ 1 & 1 & -2 & -3 \end{array}\right)$ $A\vec{x} = \vec{b}$.

$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 1 & a & 1 & -2 \\ a & 1 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & a-1 & 3 & 1 \\ 0 & 1-a & 1+2a & -1+3a \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & a-1 & 3 & 1 \\ 0 & 0 & -2+2a & 3a \end{array}\right)$

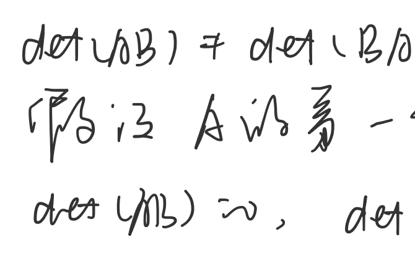
当 ① $-2+2a = 0$ 且 $3a = 0$, a 不存在.

$\therefore z = \frac{3a}{2a-2}$, $y = \frac{1}{a-1}(1 - \frac{9a}{2a-2})$　　$a \neq 1$ 且 $a \neq 0$.

$2a - 2 - 9a(a-1)$

$x = -3 + \frac{3a}{a-1} - \frac{1}{a-1}(1 - \frac{9a}{2a-2})$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3a}{2a-2} \\ \frac{-9a^2+11a-2}{2(a-1)^2} \\ \frac{-6(a-1)^2+6a(a-1)+9a}{2(a-1)^2} \end{pmatrix}$$



[illegible] $(-1, -2, -3)$.

5. $\det A = \begin{vmatrix} a & 1 & \cdots & 1 & 1 \\ 1 & a & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & a & 1 \\ 1 & 1 & \cdots & 1 & a \end{vmatrix} = \begin{vmatrix} a & 1 & \cdots & 1 & 1 \\ 1-a & a-1 & \cdots & 0 & 0 \\ 0 & 1-a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a-1 & 0 \\ 0 & 0 & \cdots & 1-a & a-1 \end{vmatrix}$

$= (a-1) \begin{vmatrix} a & 1 & \cdots & 1 \\ 1-a & a-1 & \cdots & 0 \\ 0 & 1-a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a-1 \end{vmatrix} + (-1)^{n+1}(1-a)^{n-1}$

$a(a-1) + (a-1)$

$\begin{matrix} a & 1 & 0 \\ 1-a & a-1 & 0 \\ 0 & 1-a & a-1 \end{matrix}$

$a(a-1)^2 + (a-1)^2$

$\Delta_n = (a-1)\Delta_{n-1} + (a-1)^{n-1}$.

$= (a-1)[(a-1)\Delta_{n-2} + (a-1)^{n-2}] + (a-1)^{n-1}$

$= (a-1)^2\Delta_{n-2} + 2(a-1)^{n-1}$

$= \cdots$

$= (a-1)^{n-3}\Delta_3 + (n-3)(a-1)^{n-1}$

$\Delta_1 = a$, $\Delta_2 = (a-1)(a+1)$, $\Delta_3 = (a+1)(a-1)^2$.

$\therefore \Delta_n = (a-1)^{n-3}(a+1)(a-1)^2 + (n-3)(a-1)^{n-1}$

$= \begin{cases} (a-1)^{n-1}(a+1), & n \geq 2 \\ a & n = 1. \end{cases}$

$\operatorname{rank} A = \begin{cases} 0, & a = 0 \\ 1, & a \neq 0 \text{ 且 } n = 1 \text{ 或 } a = 1 \\ n-1, & a = -1 \text{ 且 } n \geq 2 \\ n, & a \neq 0, -1, 1 \text{ 且 } n \geq 2 \end{cases}$

6. 写过.

7. (1) $A_{ij} = a_{ij}$, 代数余子式非零.

所以 $\operatorname{rank} A \geq n-1$.

且 $A_{ij} \times a_{ij} \neq 0$, 即 n 阶子式非零.

$\therefore \operatorname{rank} A = n$. A 可逆.

$\det A = \sum_{i=1}^{n} a_{1i}A_{1i} = \cdots = \sum_{i=1}^{n} a_{ni}A_{ni}$

$= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}A_{ij} = \frac{1}{n}\det(AA^*)$

$= \frac{1}{n}(\det A)(\det A^*)$

$= \frac{1}{n}|A||A|^{n-1}$　　$|A| = \frac{1}{n}|A|^n$

$\therefore \det A = \sqrt[n-1]{\frac{1}{n}}$

(2)

8. $(AB)\vec{x} = 0$ 与 $B\vec{x} = 0$ 同解.