

中国科学技术大学 2019-2020 学年第二学期
线性代数 (B1) 期末考试

- (4分 x6=24分) 填空题
 - (1) 已知实系数线性方程组 $\begin{cases} 3x_1 + 2x_2 - x_3 = 6 \\ x_1 + ax_2 + 2x_3 = 9 \\ 2x_1 - x_2 + 3x_3 = 3 \end{cases}$ 有唯一解, 则 a 满足的条件是 $\underline{\quad}$.
 - (2) 已知 3 阶方阵 $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$, 那么 $A^3 = \underline{\quad}$.
 - (3) 设向量组 $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^3$ 线性无关, 则线性子空间 $V = \text{span}\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3, 2\alpha_1 + 4\alpha_2\}$ 的维数是 $\underline{\quad}$.
 - (4) 已知线性变换 σ 在某组基下的矩阵为 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, 在另一组基下的矩阵为 $\begin{pmatrix} a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, 则 $x = \underline{\quad}$, $y = \underline{\quad}$.
 - (5) 在 \mathbb{R}^3 中, 基 $\alpha_1, \alpha_2, \alpha_3$ 的度量矩阵为 $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, 由 $\alpha_1, \alpha_2, \alpha_3$ 按原顺序 Schmidt 正交化得到的标准正交基为 $\underline{\quad}$.
 - (6) 若实二次型 $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$ 正定, 则参数 t 满足 $\underline{\quad}$.
- (5分 x4=20分) 判断题
 - (1) 已知向量组 $\alpha_1, \dots, \alpha_n$ 线性无关且可以由向量组 β_1, \dots, β_n 线性表示, 则 β_1, \dots, β_n 线性无关.
 - (2) $\forall a \in \mathbb{R}$, 方阵 $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ 与 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 相似.
 - (3) 数域 F 上 n 阶正交阵的行向量或列向量组构成 F^n 的一组标准正交基.
 - (4) 设 V 是有限个 n 阶实方阵全体构成的集合, 它在加法法和数乘下构成一个 n 维实线性空间, 那么 V 中对称阵全体构成它的一个 n 维子空间.
- (12分) 设某个 4 元线性方程组的系数矩阵为 A , 满足 $\text{rank} A = 3$. 已知 $\alpha_1, \alpha_2, \alpha_3$ 是它的 3 个解, 其中 $\alpha_1 = (1, -2, -3, 4)^T, \alpha_2 = (2, 0, 2, 0)^T, \alpha_3 = (0, 1, 2)^T, \alpha_4 = (1, 0, 0)^T$ 分别称为 $\beta_1 = (1, 2, 0)^T, \beta_2 = (2, 0, 1)^T, \beta_3 = (3, 0, 1)^T$. 求:
 - (1) 证明: 这个线性方程组是非齐次的. (2) 求出这个线性方程组的通解.
- (14分) 用初等变换法求矩阵 A 的逆与行列式, 其中 $A = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ -1 & 0 & 3 & \dots & n-1 & n \\ 1 & -2 & 0 & \dots & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \dots & 0 & n \\ 1 & 2 & 3 & \dots & 0 & n \end{pmatrix}$.
- (14分) \mathbb{R}^3 上线性变换 σ 把 $\alpha_1 = (2, 3, 5)^T, \alpha_2 = (0, 1, 2)^T, \alpha_3 = (1, 0, 0)^T$ 分别映为 $\beta_1 = (1, 2, 0)^T, \beta_2 = (2, 4, -1)^T, \beta_3 = (3, 0, 5)^T$. 求:
 - (1) σ 在基 $\alpha_1, \alpha_2, \alpha_3$ 下的矩阵 A .
 - (2) σ 在自然基下的矩阵 B .
- (16分) 设实二次型 $Q(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 - 8x_1x_3 - 4x_2x_3$.
 - (1) 利用正交变换将该二次型化为标准型, 并写出相应的正交变换矩阵.
 - (2) 判断 $Q(x_1, x_2, x_3) = 1$ 在三维直角坐标系中所表示的曲面的类型.

1. (1) $A \neq -\frac{9}{1}$ ✓

$$\begin{pmatrix} 3 & 2 & -1 & b \\ 1 & a & 2 & 9 \\ 2 & -1 & 3 & 3 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} 1 & a & 2 & 9 \\ 3 & 2 & -1 & b \\ 2 & -1 & 3 & 3 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} 1 & a & 2 & 9 \\ 0 & 2-3a & -7 & b-27 \\ 0 & -1 & -1 & -15 \end{pmatrix}$$

(2) $9b$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ ✓

$$A \approx \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \approx \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|1 \ 2 \ 3| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \times 4 + 9 = 14$$

$$A^3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} 14 \times 14 \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

(3) 2 ✓

$$\lambda(\alpha_1 + \alpha_2 + \alpha_3) = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3$$

$$= (\lambda_1 + \lambda_2 + \lambda_3)\alpha_1 + (\lambda_1 + 2\lambda_2 + 3\lambda_3)\alpha_2 + (\lambda_1 + 3\lambda_2 + 4\lambda_3)\alpha_3 = 0$$

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \\ \lambda_1 + 3\lambda_2 + 4\lambda_3 = 0 \end{cases} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

(4) 0 ✓

$$1+x = 4 - (1+x), \quad 3+x = 4$$

$$-1 = -4, \quad x = 1$$

(5) $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$ ✓

$$|\vec{\alpha}_1| = 1, |\vec{\alpha}_2| = \sqrt{2}, |\vec{\alpha}_3| = \sqrt{6}$$

$$(\vec{\alpha}_1, \vec{\alpha}_2) = 0, \quad (\vec{\alpha}_1, \vec{\alpha}_3) = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}, \quad (\vec{\alpha}_2, \vec{\alpha}_3) = 1$$

$$\vec{\beta}_3 = \vec{\alpha}_3 - (\vec{\alpha}_1, \vec{\alpha}_3)\vec{\alpha}_1 - (\vec{\alpha}_2, \vec{\alpha}_3)\vec{\alpha}_2$$

$$= \vec{\alpha}_3 - \sqrt{3}\vec{\alpha}_1 - \vec{\alpha}_2 = \vec{\alpha}_3 - \vec{\alpha}_2 - \sqrt{3}\vec{\alpha}_1$$

$$(\vec{\alpha}_3 - \vec{\alpha}_2, \vec{\alpha}_3 - \vec{\alpha}_2) = |\vec{\alpha}_3|^2 + |\vec{\alpha}_2|^2 - 2(\vec{\alpha}_3, \vec{\alpha}_2)$$

$$= 6 + 2 - 2 = 6$$

$$\vec{\alpha}_3 = \vec{\alpha}_2 - \vec{\alpha}_1$$

(6) $-\sqrt{3} + 1 < t < \sqrt{3} + 1$ ✓

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & t & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & t-1 \\ 0 & t-1 & b-1 \end{pmatrix}$$

$$2 - (t-1)^2 > 0 \Rightarrow 1 < t < 3$$

2. (1) $\vec{v} = k\vec{a} + l\vec{b}, \vec{v} = m\vec{a} + n\vec{b} \Rightarrow k\vec{a} + l\vec{b} = m\vec{a} + n\vec{b}$

(2) $\vec{v} = \vec{a}$ ✓

$$P(\lambda) = \begin{vmatrix} \lambda-1 & -4 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2$$

$$\lambda=1, \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(3) $\vec{v} = \vec{a}$ ✓

(4) $\vec{v} = \vec{a}$ ✓

3. $A\vec{x} = \vec{b}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} \end{pmatrix} = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4)$$

因为 \vec{v}_1 奇数, $n=4$ 且 $\text{rank} A = 3$, 所以 \vec{v}_1 为 \vec{b} 的线性组合.

$$\vec{v}_1 = 5\vec{a}_2 - 2\vec{a}_3 - 3\vec{a}_1$$

$$\vec{v}_2 = 5\vec{a}_2 - 2\vec{a}_3 - 3\vec{a}_1 = \vec{b}$$

$$\vec{v}_3 = 5\vec{a}_2 - 2\vec{a}_3 - 3\vec{a}_1 = \vec{b}$$

$$\vec{v}_4 = 5\vec{a}_2 - 2\vec{a}_3 - 3\vec{a}_1 = \vec{b}$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 2 & 1 & -2 & -2 & -2 \\ 3 & 2 & 0 & 1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$|\vec{b}| = 1$ ✓

$$A(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3)$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix}$$

$$A^{-1} \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 7 \\ 0 & -5 & 3 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 \\ 5 & 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 \\ 5 & 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 2 & -1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -7 & -5 \\ -10 & -23 & -15 \\ -3 & -24 & -7 \end{pmatrix}$$

$$A(\vec{e}_1, \vec{e}_2, \vec{e}_3) = \vec{b}$$

$$= A(\alpha_1, \alpha_2, \alpha_3) (\alpha_1, \alpha_2, \alpha_3)^T$$

$$= (\alpha_1, \alpha_2, \alpha_3) A^T$$

$$= (\vec{e}_1, \vec{e}_2, \vec{e}_3) T^{-1} A^T$$

$$\therefore B = T^{-1} A^T = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}$$

$$B = (\lambda_1, \lambda_2, \lambda_3) \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & -2 \\ -4 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$P(\lambda) = \begin{vmatrix} \lambda-1 & 2 & 4 \\ 2 & \lambda-4 & 2 \\ 4 & 2 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda-4)(\lambda-1) + 32 - 16(\lambda-4) - 8(\lambda-1)$$

$$= (\lambda-1)(\lambda^2 - 6\lambda + 8) = (\lambda-1)(\lambda-2)(\lambda-4)$$

$$\lambda=1, \begin{pmatrix} 0 & 2 & 4 \\ 2 & -3 & 2 \\ 4 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda=2, \begin{pmatrix} 1 & 2 & 4 \\ -2 & -2 & 2 \\ -4 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda=4, \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}$$