18～19春期末

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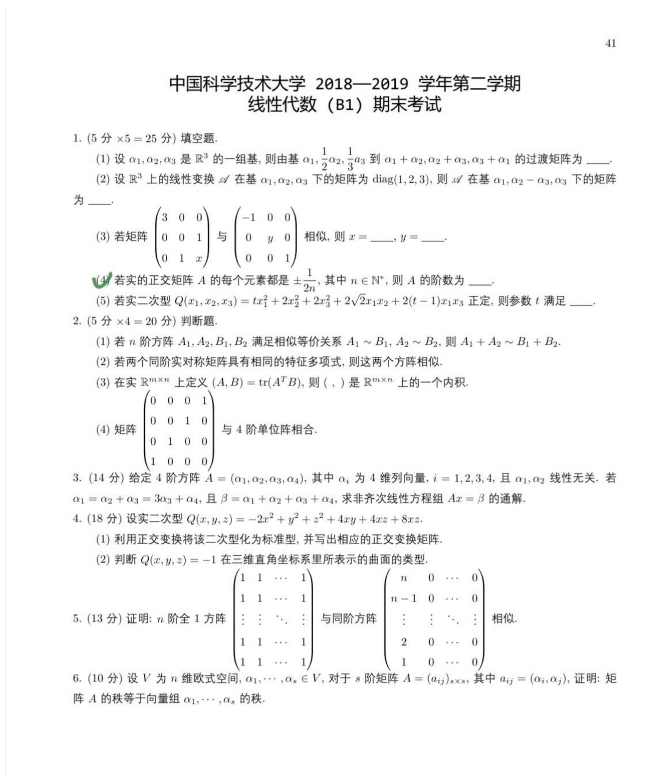
线性变换…
方程组…



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中国科学技术大学 2018—2019 学年第二学期
线性代数（B1）期末考试

1.（5 分 ×5 = 25 分）填空题.

(1) 设 $\alpha_1,\alpha_2,\alpha_3$ 是 $\mathbb{R}^3$ 的一组基，则由基 $\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3$ 到 $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$ 的过渡矩阵为 ____.

(2) 设 $\mathbb{R}^3$ 上的线性变换 $\mathscr{A}$ 在基 $\alpha_1,\alpha_2,\alpha_3$ 下的矩阵为 $\mathrm{diag}(1,2,3)$，则 $\mathscr{A}$ 在基 $\alpha_1,\alpha_2-\alpha_3,\alpha_3$ 下的矩阵为 ____.

(3) 若矩阵 $\begin{pmatrix}3&0&0\\0&0&1\\0&1&x\end{pmatrix}$ 与 $\begin{pmatrix}-1&0&0\\0&y&0\\0&0&1\end{pmatrix}$ 相似，则 $x=$ ____，$y=$ ____.

(4) 若实的正交矩阵 A 的每个元素都是 $\pm\frac{1}{2n}$，其中 $n\in\mathbb{N}^*$，则 A 的阶数为 ____.

(5) 若实二次型 $Q(x_1,x_2,x_3)=tx_1^2+2x_2^2+2x_3^2+2\sqrt{2}x_1x_2+2(t-1)x_1x_3$ 正定，则参数 t 满足 ____.

2.（5 分 ×4 = 20 分）判断题.

(1) 若 n 阶方阵 A_1,A_2,B_1,B_2 满足相似等价关系 $A_1\sim B_1$, $A_2\sim B_2$，则 $A_1+A_2\sim B_1+B_2$.

(2) 若两个同阶实对称矩阵具有相同的特征多项式，则这两个方阵相似.

(3) 在实 $\mathbb{R}^{n\times n}$ 上定义 $(A,B)=\mathrm{tr}(A^TB)$，则 $(\ ,\ )$ 是 $\mathbb{R}^{n\times n}$ 上的一个内积.

(4) 矩阵 $\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$ 与 4 阶单位阵相合.

3.（14 分）给定 4 阶方阵 $A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$，其中 α_i 为 4 维列向量，$i=1,2,3,4$，且 α_1,α_2 线性无关．若 $\alpha_1=\alpha_2+\alpha_3=3\alpha_3+\alpha_4$，且 $\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4$，求非齐次线性方程组 $Ax=\beta$ 的通解.

4.（18 分）设实二次型 $Q(x,y,z)=-2x^2+y^2+z^2+4xy+4xz+8yz$.

(1) 利用正交变换将该二次型化为标准型，并写出相应的正交变换矩阵.

(2) 判断 $Q(x,y,z)=-1$ 在三维直角坐标系里所表示的曲面的类型.

5.（13 分）证明：n 阶全 1 方阵 $\begin{pmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&1\\1&1&\cdots&1\end{pmatrix}$ 与同阶方阵 $\begin{pmatrix}n&0&\cdots&0\\n-1&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\2&0&\cdots&0\\1&0&\cdots&0\end{pmatrix}$ 相似.

6.（10 分）设 V 为 n 维欧式空间，$\alpha_1,\cdots,\alpha_s\in V$，对于 s 阶矩阵 $A=(a_{ij})_{s\times s}$，其中 $a_{ij}=(\alpha_i,\alpha_j)$，证明：矩阵 A 的秩等于向量组 $\alpha_1,\cdots,\alpha_s$ 的秩.

1. (1) $\begin{pmatrix}1&0&1\\2&2&0\\0&3&3\end{pmatrix}$ ✓

$(\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&\\&\frac{1}{2}&\\&&\frac{1}{3}\end{pmatrix}$

$(\alpha_1,\alpha_2,\alpha_3)=(\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3)\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}$

$(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&1\\1&1&\\&1&1\end{pmatrix}$

$=(\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3)\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}\begin{pmatrix}1&&1\\1&1&\\&1&1\end{pmatrix}$

(2) $\begin{pmatrix}1&0&0\\0&2&0\\0&-1&3\end{pmatrix}$ ✓

$\mathscr{A}(\alpha_1,\alpha_2,\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}=AM$.

$(\alpha_1,\alpha_2-\alpha_3,\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&\\&1&\\&-1&1\end{pmatrix}$，$\tilde{A}=AT$.

$\mathscr{A}(\alpha_1,\alpha_2-\alpha_3,\alpha_3)=(\alpha_1,\alpha_2-\alpha_3,\alpha_3)M'=\tilde{A}M'$

$\parallel\qquad=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&\\&1&\\&-1&1\end{pmatrix}M'=ATM'$

$T\mathscr{A}(\alpha_1,\alpha_2,\alpha_3)$

$M'=T^{-1}MT=\begin{pmatrix}1&&\\&1&\\&1&1\end{pmatrix}\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&-1&1\end{pmatrix}$

$=\begin{pmatrix}1&0&0\\0&2&0\\0&2&3\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&-1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&2&0\\0&-1&3\end{pmatrix}$

(3) 0，3 ✓

$p_\lambda(A)=\begin{vmatrix}\lambda-3&&\\&\lambda&-1\\&-1&\lambda-x\end{vmatrix}=(\lambda-3)\lambda(\lambda-x)-(\lambda-3)$
$=(\lambda-3)(\lambda^2-x\lambda-1)$

$p_\lambda(B)=\begin{vmatrix}\lambda+1&&\\&\lambda-y&\\&&\lambda-1\end{vmatrix}$．$-1,y,1$．$\Rightarrow y=3,\ x=0$

(4) 4 ✓

$\begin{pmatrix}\frac{1}{2n}\\\vdots\\\frac{1}{2n}\end{pmatrix}n$个．$\sqrt{(\frac{1}{2n})^2\times n}=1$，$\sqrt{\frac{1}{4n}}=1$．$16n^2=1$.

(5) $1<t<3$. ✓

$Q=(x_1\ x_2\ x_3)\begin{pmatrix}t&\sqrt{2}&t-1\\\sqrt{2}&2&0\\t-1&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$2t-2>0$，$t>1$.

$4t-2(t-1)^2-4>0$．$2t-t^2+2t-1-2=-t^2+4t-3>0$

$t^2-4t+3<0$，$(t-3)(t-1)<0$．$1<t<3$.

2. (1) 错误. ✓　反例．$A_1=\begin{pmatrix}1&\\&0\end{pmatrix}$．$B_1=A_1=A_2$，$B_2=\begin{pmatrix}0&\\&1\end{pmatrix}$
$A_1+A_2=\begin{pmatrix}2&\\&0\end{pmatrix}$　$B_1+B_2=\begin{pmatrix}1&\\&1\end{pmatrix}$

$\exists P_1$，$P_1^{-1}A_1P_1=B_1$　　$B_1+B_2=P_1^{-1}A_1P_1+P_2^{-1}A_2P_2$

$\exists P_2$，$P_2^{-1}A_2P_2=B_2$.　　$P_1=P_2$时，

$B_1+B_2=P^{-1}(A_1+A_2)P$.

(2) 正确. ✓

$p_A(\lambda)=\lambda^n-\mathrm{tr}(A)\lambda^{n-1}-\cdots-|A|$．　实对称→正交相似

特征值相同．都可相似于对角阵．　相似的对角化.

(3) 正确 ✓

$A=(\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_n)$．$B=(\vec{\beta}_1,\vec{\beta}_2,\cdots,\vec{\beta}_n)$

$A^TB=\begin{pmatrix}\vec{\alpha}_1^T\\\vdots\\\vec{\alpha}_n^T\end{pmatrix}(\vec{\beta}_1\cdots\vec{\beta}_n)=(\vec{\alpha}_i\cdot\vec{\beta}_j)_{n\times n}$.

$\mathrm{tr}(A^TB)=\vec{\alpha}_1\cdot\vec{\beta}_1+\cdots+\vec{\alpha}_n\cdot\vec{\beta}_n$.

对称性：$(B,A)=\mathrm{tr}(B^TA)=\sum_{i=1}^n\vec{\beta}_i\cdot\vec{\alpha}_i=\sum_{i=1}^n\vec{\alpha}_i\cdot\vec{\beta}_i=(A,B)$

线性性：$(\lambda A,B)=\mathrm{tr}((\lambda A)^TB)=\sum_{i=1}^n\lambda\vec{\alpha}_i\cdot\vec{\beta}_i=\lambda\sum_{i=1}^n\vec{\alpha}_i\cdot\vec{\beta}_i=\lambda(A,B)$

$(A+C,B)=\mathrm{tr}((A+C)^TB)=\mathrm{tr}(A^TB+C^TB)$
$=\sum_{i=1}^n(\vec{\alpha}_i+\vec{\gamma}_i)\cdot\vec{\beta}_i=\sum_{i=1}^n\vec{\alpha}_i\cdot\vec{\beta}_i+\sum_{i=1}^n\vec{\gamma}_i\cdot\vec{\beta}_i$
$=(A,B)+(C,B)$. ✓

正定性：$(A,A)=\sum_{i=1}^n\vec{\alpha}_i^2\geq0$.
取0时当且仅当所有$\vec{\alpha}_i=\vec{0}$．$A=0$. ✓

∴满足内积.

(4) 错误. ✓

$p_A(\lambda)=\begin{vmatrix}\lambda&&&-1\\&\lambda&-1&\\&-1&\lambda&\\-1&&&\lambda\end{vmatrix}=\lambda\begin{vmatrix}\lambda&-1&\\-1&\lambda&\\&&\lambda\end{vmatrix}+(-1)(-1)^{1+4}\begin{vmatrix}&\lambda&-1\\&-1&\lambda\\-1&&\end{vmatrix}$

$=\lambda(\lambda^3-\lambda)+1-\lambda^2$

$=\lambda^2(\lambda^2-1)-(\lambda^2-1)=(\lambda^2-1)^2$

$=(\lambda+1)^2(\lambda-1)^2$.

特征值 $1,1,-1,-1$，相合于 $\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}$

3. $A=(\vec{\alpha}_1,\vec{\alpha}_2,\vec{\alpha}_3,\vec{\alpha}_4)$.

$\vec{\alpha}_3=\vec{\alpha}_1-\vec{\alpha}_2$,

$\vec{\alpha}_4=\vec{\alpha}_1-3\vec{\alpha}_3=\vec{\alpha}_1-3\vec{\alpha}_1+3\vec{\alpha}_2=-2\vec{\alpha}_1+3\vec{\alpha}_2$.

$\mathrm{rk}(A)=2$，$\vec{\alpha}_1,\vec{\alpha}_2$为极大无关组.

$A=(\vec{\alpha}_1,\vec{\alpha}_2,\vec{\alpha}_3,\vec{\alpha}_4)=(\vec{\alpha}_1,\vec{\alpha}_2,\vec{\alpha}_1-\vec{\alpha}_2,-2\vec{\alpha}_1+3\vec{\alpha}_2)$

$=(\vec{\alpha}_1,\vec{\alpha}_2)\begin{pmatrix}1&&1&-2\\&1&-1&3\end{pmatrix}$.

$\vec{\beta}=\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3+\vec{\alpha}_4=\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_1-\vec{\alpha}_2-2\vec{\alpha}_1+3\vec{\alpha}_2=3\vec{\alpha}_2$.

$A\vec{x}=\vec{\beta}$，$(\vec{\alpha}_1,\ \vec{\alpha}_2,\ \vec{\alpha}_1-\vec{\alpha}_2,\ -2\vec{\alpha}_1+3\vec{\alpha}_2)\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=(3\vec{\alpha}_2)$

$x_1\vec{\alpha}_1+x_2\vec{\alpha}_2+x_3\vec{\alpha}_1-x_3\vec{\alpha}_2-2x_4\vec{\alpha}_1+3x_4\vec{\alpha}_2=3\vec{\alpha}_2$,

$(x_1+x_3-2x_4)\vec{\alpha}_1+(x_2-x_3+3x_4-3)\vec{\alpha}_2=0$.

$\begin{cases}x_1+x_3-2x_4=0\\x_2-x_3+3x_4=3\end{cases}\Rightarrow\begin{pmatrix}1&0&1&-2&0\\0&1&-1&3&3\end{pmatrix}$

$x_4=t_1$，$x_3=t_2$．$x_2=3-3t_1+t_2$

$x_1=2t_1-t_2$

$\therefore\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}t_1+\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}t_2+\begin{pmatrix}0\\3\\0\\0\end{pmatrix}$ ✓

4. $Q=(x\ y\ z)\begin{pmatrix}-2&2&2\\2&1&4\\2&4&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$　$A=\begin{pmatrix}-2&2&2\\2&1&4\\2&4&1\end{pmatrix}$

$p_A(\lambda)=\begin{vmatrix}\lambda+2&-2&-2\\-2&\lambda-1&-4\\-2&-4&\lambda-1\end{vmatrix}=(\lambda+2)(\lambda-1)^2-16-16$
$-4(\lambda-1)-4(\lambda-1)-16(\lambda+2)$

$=(\lambda+2)(\lambda^2-2\lambda+1)-32-8\lambda+8-16\lambda-32$

$=\lambda^3-3\lambda+2-24\lambda-56$

$=\lambda^3-27\lambda-54$

$=(\lambda+3)(\lambda-6)(\lambda+3)$

$$\begin{array}{r|l}&\lambda^2-3\lambda-18\\\hline\lambda+3&\lambda^3+0\lambda^2-27\lambda-54\\&\lambda^3+3\lambda^2\\\hline&-3\lambda^2-27\lambda\\&-3\lambda^2-9\lambda\\\hline&-18\lambda-54\\&-18\lambda-54\end{array}$$

$\lambda=6$，$\begin{pmatrix}8&-2&-2\\-2&5&-4\\-2&-4&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$\begin{pmatrix}0&-18&18\\0&9&-9\\-2&-4&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$　$-2x_1-x_3=0$

$x_2=x_3$，$-2x_1-4x_3+5x_3=0$，$x_1=\frac{1}{2}x_3$

∴特征向量 $\vec{x}_3=(1,2,2)^T$．单位化 $\vec{e}_3=(\frac{1}{3},\frac{2}{3},\frac{2}{3})^T$

$\lambda=-3$，$\begin{pmatrix}-1&-2&-2\\-2&-4&-4\\-2&-4&-4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$x_1+2x_2+2x_3=0$.

得一组基 $\vec{x}_1=(-2,0,1)^T$．$\vec{x}_2=(-2,1,0)^T$.

$\vec{e}_1=\frac{1}{\sqrt{5}}(-2,0,1)^T$，　$4+16+25=45$　$\frac{3}{5}\sqrt{5}$
$\frac{1}{5}[(-10,5,0)-(-8,0,4)]$

$\vec{\beta}_2=\vec{x}_2-(\vec{e}_1,\vec{x}_2)\vec{e}_1=(-2,1,0)-\frac{4}{5}(-2,0,1)^T$

$=\frac{1}{5}(-2,5,-4)^T$.

$\vec{e}_2=\frac{1}{15\sqrt{5}}(-2,5,-4)^T$ ✓

$\therefore P^TAP=\begin{pmatrix}-3&&\\&-3&\\&&6\end{pmatrix}$．$P=\begin{pmatrix}-\frac{2}{\sqrt{5}}&-\frac{2}{15\sqrt{5}}&\frac{1}{3}\\0&\frac{5}{15\sqrt{5}}&\frac{2}{3}\\\frac{1}{\sqrt{5}}&-\frac{4}{15\sqrt{5}}&\frac{2}{3}\end{pmatrix}$ ✓

(2) 正交变换后，$-3x^2-3y^2+6z^2=-1$,

$3x^2+3y^2-6z^2=1$，单叶双曲面.

5. 对 $\begin{pmatrix}n&\cdots&0\\n-1&\cdots&0\\\vdots&\ddots&\vdots\\2&\cdots&0\\1&\cdots&0\end{pmatrix}\triangleq B$，$\mathrm{rk}(B)=1$，特征值为 $0,n$，

0为 $n-1$ 重特征值.

对 $\begin{pmatrix}1&\cdots&1\\1&\cdots&1\\\vdots&\ddots&\vdots\\1&\cdots&1\end{pmatrix}\triangleq A$，$\mathrm{rk}(A)=\mathrm{rk}\begin{pmatrix}1&0&\cdots&0\\1&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\1&0&\cdots&0\end{pmatrix}=1$，0为特征值.

$\begin{pmatrix}-1&-1&\cdots&-1\\-1&-1&\cdots&-1\\\vdots&\vdots&\ddots&\vdots\\-1&-1&\cdots&-1\end{pmatrix}\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}0\\\vdots\\0\end{pmatrix}$，$x_1+\cdots+x_n=0$.
特征子空间为 $n-1$ 重.

$\begin{pmatrix}1&\cdots&1\\\vdots&&\vdots\\1&\cdots&1\end{pmatrix}\vec{x}=n\vec{x}$.

6. $A=\begin{pmatrix}(\alpha_1,\alpha_1)&\cdots&(\alpha_1,\alpha_n)\\\vdots&&\vdots\\(\alpha_n,\alpha_1)&\cdots&(\alpha_n,\alpha_n)\end{pmatrix}$

$\mathscr{A}(x,y,z)=(x,y,0)$ 在 e^3 下的矩阵.

$\mathscr{A}(\vec{e}_1)=\mathscr{A}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$　$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$.

$\mathscr{A}(\vec{e}_2)=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\mathscr{A}(\vec{e}_3)=\begin{pmatrix}0\\0\\0\end{pmatrix}$

微分变换在 $P_n[x]$

$\mathscr{A}(1,x,\cdots,x^{n-1})=(0,1,2x,\cdots,(n-1)x^{n-2})$

$=(1,x,x^2,\cdots,x^{n-1})\begin{pmatrix}0&1&0&\cdots&0\\0&0&2&\cdots&0\\0&0&0&\ddots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\&&&&n-1\\0&0&0&\cdots&0\end{pmatrix}$

$\alpha\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}\xrightarrow{\mathscr{A}}\begin{pmatrix}2&1&0\\3&0&1\\5&0&-1\end{pmatrix}$

$\mathscr{A}(\alpha_1\ \alpha_2\ \alpha_3)=(\alpha_1\ \alpha_2\ \alpha_3)A=(\beta_1\ \beta_2\ \beta_3)$

在α下：$A=(\alpha)^{-1}(\beta)$

$\mathscr{A}(e_1\ e_2\ e_3)=\mathscr{A}((\alpha_1\ \alpha_2\ \alpha_3)\cdot\alpha^{-1})$

$=\mathscr{A}(\alpha_1\ \alpha_2\ \alpha_3)(\alpha_1\ \alpha_2\ \alpha_3)^{-1}$

$=(e_1,e_2,e_3)(\beta_1\ \beta_2\ \beta_3)(\alpha_1\ \alpha_2\ \alpha_3)^{-1}$

在e下：$\tilde{A}=\beta\alpha^{-1}$　　$\alpha A\alpha^{-1}$
$=(\alpha^{-1})^{-1}A\alpha^{-1}$

$A=aE_{11}+bE_{12}+cE_{21}+dE_{22}$

$=(E_{11}\ E_{12}\ E_{21}\ E_{22})\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$(\alpha)=\begin{pmatrix}1&2&1\\2&1&0\\3&0&0\end{pmatrix}$．$(\beta)=\begin{pmatrix}1&2&4\\1&1&3\\2&3&8\end{pmatrix}$

$(\beta)=(\alpha)T$，$T=(\alpha)^{-1}(\beta)$　$(\alpha)=(\beta)T^{-1}$

$\alpha=(\alpha_1\ \alpha_2\ \alpha_3)\begin{pmatrix}1\\3\\-4\end{pmatrix}=(\beta)T^{-1}\begin{pmatrix}1\\3\\-4\end{pmatrix}$．$T^{-1}\begin{pmatrix}1\\3\\-4\end{pmatrix}$

$A^2=A$，A可对角化.

度量矩阵G．$g_{ij}=(\vec{\alpha}_i,\vec{\alpha}_j)$.

$\vec{\alpha}_1\cdots\vec{\alpha}_m$线性无关 $\Leftrightarrow\det G\neq0$.

反证.

相关．$\lambda_i\vec{\alpha}_i=0$，$\lambda_i\neq0$.

取内积，线性性．$g_{ij}x\cdots=0$ 非零解.

欧氏空间的一组基 $S=\{\alpha_1\cdots\alpha_n\}$.

→标准正交基 $(\varepsilon_1\cdots\varepsilon_n)=(\alpha_1\cdots\alpha_n)P$.

P是可逆上三角阵.

$\forall$：S的度量矩阵G

→ $P^TGP=I$（相合）.

$S.T$都是标准正交基，度量矩阵都是I.

$S=TP$．$P^TIP=I$，$P^T=P^{-1}$.

正交方阵P.