20～21期中

2021年11月30日 星期二　　上午9:14

中国科学技术大学 2020—2021 学年第二学期
线性代数（B1）期中考试

1. (5 分 ×5 = 25 分) 填空题.

(1) 设三维欧式空间中有向量 a, b, c, 它们的模长相同且两两所成夹角相等. 已知在空间直角坐标系下 $a = (1,1,0)$, $b = (0,1,1)$, $c =$ ____.

(2) 设 $A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, 则 $A^* =$ ____.

(3) 设 $A = \begin{pmatrix} 1 & 0 & 1 & -4 \\ 1 & 3 & 4 & 2 \\ 2 & 1 & 4 & 4 \\ 2 & 3 & -3 & 2 \end{pmatrix}$, M_{ij} 为余子式, 则 $M_{31} - M_{32} + M_{33} =$ ____.

(4) 已知向量组 $\alpha_1 = (1,2,3,4)$, $\alpha_2 = (2,3,4,5)$, $\alpha_3 = (3,4,5,6)$, $\alpha_4 = (4,5,6,k)$, 且 $\operatorname{rank}\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} = 2$, 则 $k =$ ____.

(5) 设 $\mathbb{P}_3[x]$ 为实数域 $\mathbb{R}$ 上次数不超过 3 的多项式全体, 则基 $\{1, x, x(x-1), x(x-1)(x-2)\}$ 到自然基 $\{1, x, x^2, x^3\}$ 的过渡矩阵 $T =$ ____.

2. (5 分 ×5 = 25 分) 判断题.

(1) 设 $A = \begin{pmatrix} 2 & 4 & -3 \\ -1 & 3 & 2 \\ 7 & -1 & -12 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 2 & -4 & 3 \end{pmatrix}$, 则 A 与 B 不相抵.

(2) 设 A, B 为 n 阶方阵, 则 $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|$.

(3) 向量组 $\alpha_1, \cdots, \alpha_n$ 线性相关的充要条件是向量组中的任意一个向量 α_i 都可以由剩余的 $n-1$ 个向量线性表示.

(4) 设 A, B 为满足 $AB = 0$ 的任意两个非零矩阵, 则必有 A 的列向量线性相关, B 的行向量线性相关.

(5) 设 A 为 n 阶非零实方阵, 若 $A^T = A^*$, 则 A 可逆.

3. (12 分) 当 λ 取何值时, 线性方程组 $\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 7x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$ 有解, 并求处它的通解.

4. (14 分) 设 n 阶方阵 $A = \begin{pmatrix} 1+a & 1 & \cdots & 1 \\ 1 & 1+a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a \end{pmatrix}$, 其中 $a > 0$, 求 $\det A$ 及 A^{-1}.

5. (14 分) 设矩阵 $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, 令 V 是所有与 A 可交换的三阶实对称方阵全体.

(1) 证明: V 按矩阵的加法与数乘运算构成的实数域 $\mathbb{R}$ 上的线性空间.

(2) 求 V 的维数与一组基.

6. (10 分) 设 $A \in F^{m\times n}$, $B \in F^{n\times p}$. 证明: $\operatorname{rank}(AB) = \operatorname{rank} B$ 成立的充要条件是方程组 $ABx = 0$ 的解均为方程组 $Bx = 0$ 的解.

T2 (1) A不可逆 $\Leftrightarrow \exists B \neq 0$, 使 $AB = 0$.

正确. A不可逆 $\Leftrightarrow Ax = 0$ 有非零解.

$\Leftrightarrow \exists x \neq 0$ 为列向量 使 $A(x\,0) = 0$

(2) $\vec{b}$ 可被 $\{\vec{\alpha}_1, \cdots \vec{\alpha}_s\}$ 线性表示, 但不能被 $\{\vec{\alpha}_1, \cdots \vec{\alpha}_{s-1}\}$ 线性表示 $\Rightarrow$ $\vec{\alpha}_s$ 能被 $\{\vec{b}, \vec{\alpha}_1 \cdots \vec{\alpha}_{s-1}\}$ 线性表示.

正确. $\vec{b} = \sum_{i=1}^{s} \lambda_i \alpha_i$, $\vec{\alpha}_s = \_\ \_\ \ \_$ ✓.

(4). 矩阵 A.B 满足 $AB - BA = A \Rightarrow$ 不可逆.

正确. 假设 A可逆, 则 $B - A^{-1}BA = I$.

$\therefore n = tr(I) = tr(B) - tr(A^{-1}BA) = tr(B) - tr(B) = 0$ 矛盾.

1.(1) $(1, 0, 1)$.

(2) $AA^* = \det(A) I$. $A^{-1} = \frac{A^*}{\det A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$

$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$

$\det A = \frac{5}{4} - \frac{1}{4} - \frac{1}{2} = \frac{3}{2}$. $A^* = \frac{3}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{9}{2} \end{pmatrix}$

(3) $A = \begin{pmatrix} 1 & 0 & 1 & -4 \\ 1 & 3 & 4 & 2 \\ 2 & 1 & 4 & 4 \\ 2 & 3 & -3 & 2 \end{pmatrix}$.

$M_{31} = \begin{vmatrix} 0 & 1 & -4 \\ 3 & 4 & 2 \\ 3 & -3 & 2 \end{vmatrix} = -1\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 4 \\ 3 & -3 \end{vmatrix} = -1 \times 0 - 4 \times (-9-12) = 81$

$M_{32} = \begin{vmatrix} 1 & 1 & -4 \\ 1 & 4 & 2 \\ 2 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ -3 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix}$
$= 8 + 6 - (2-4) - 4(-3-8)$
$= 14 + 2 + 4 \times 11 = 60$

$M_{33} = \begin{vmatrix} 1 & 0 & -4 \\ 1 & 3 & 2 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 12$

$M_{31} - M_{32} + M_{33} = 81 - 60 + 12 = 33$

(4) $\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & k \end{pmatrix} = 2$. $k = 3$.

$\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & k \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & k \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(5) $\{1, x, x^2-x, x^3-3x^2+2x\}$.

$(1\ \ x\ \ x^2-x\ \ x^3-3x^2+2x)\,T = (1\ \ x\ \ x^2\ \ x^3)$

$(1\ \ x\ \ x^2\ \ x^3)\,T^{-1} = (1\ \ x\ \ x^2-x\ \ x^3-3x^2+2x)$

$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$ $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. (1) $A = \begin{pmatrix} 2 & 4 & -3 \\ -1 & 3 & 2 \\ 7 & -1 & -12 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 2 \\ 2 & 4 & -3 \\ 7 & -1 & -12 \end{pmatrix}$

$\to \begin{pmatrix} -1 & 3 & 2 \\ 0 & 10 & 1 \\ 0 & 20 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 2 \\ 0 & 10 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $r = 2$.

$B = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -10 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $r = 2$. ✗

(2) $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A & B \\ 0 & A-B \end{vmatrix} = |A|\,|A - BA^{-1}B|$

A 可逆时.

$B - A \times (A^{-1} \times B) = 0$

$A - B \times (A^{-1} \times B) = A - BA^{-1}B$

$A^2 - ABA^{-1}B$ ✗

(3) ✗ $\exists \alpha_i$.

(4) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✗

(5) $\det A^T = \det A = \det A^*$. ✓

假设 A不可逆. $\det A = 0 = \det A^*$.

对 $\forall a_{ij}$. 有 $a_{ji} = A_{ij} = (-1)^{i+j} M_{ij}$

$\det A = \sum_{1\le i,j\le n} a_{ij} A_{ij} = \sum_{1\le i,j\le n} a_{ij} a_{ji}$

$\det A^* = \sum_{1\le i,j\le n} A_{ij} D_{ij} = \sum_{1\le i,j\le n} a_{ji} D_{ij}$

$A^* = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ & A_{ij} & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$ D_{ij} 为 A_{ij} 的代余, $n-1$ 阶.

$a_{ij} = D_{ij}$

$a_{ij} \neq 0$, $\det A = \det A^* = 0$, $D_{ij} \neq 0$. 矛盾.

3. $\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 7x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$ $\begin{pmatrix} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 7 & -4 & 11 & \lambda \end{pmatrix}$

$\to \begin{pmatrix} 2 & -1 & 1 & 1 & 1 \\ 2 & 4 & -2 & 8 & 4 \\ 2 & 14 & -8 & 22 & 2\lambda \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 & 1 & 1 \\ 0 & 5 & -3 & 7 & 3 \\ 0 & 15 & -9 & 21 & 2\lambda-1 \end{pmatrix}$

$\to \begin{pmatrix} 2 & -1 & 1 & 1 & 1 \\ 0 & 5 & -3 & 7 & 3 \\ 0 & 0 & 0 & 0 & 2\lambda-10 \end{pmatrix}$ $2\lambda - 10 \neq 0$, $\lambda \neq 5$, 无解.
$\lambda = 5$, 有解.

取 $x_3 = t_1$, $x_4 = t_2$, 有 $x_2 = \frac{3}{5} - \frac{7}{5}t_2 + \frac{3}{5}t_1$

$x_1 = \frac{1}{2} - \frac{t_1}{2} - \frac{t_2}{2} + \frac{3}{10} - \frac{7}{10}t_2 + \frac{3}{10}t_1 = \frac{4}{5} - \frac{1}{5}t_1 - \frac{6}{5}t_2$.

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} -\frac{6}{5} \\ -\frac{7}{5} \\ 0 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \\ 0 \end{pmatrix}$

4. $A = \begin{pmatrix} 1+a & 1 & \cdots & 1 \\ 1 & 1+a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a \end{pmatrix}$

$\det A = \begin{vmatrix} 1+a & 1 & \cdots & 1 \\ 1 & 1+a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a \end{vmatrix} = \begin{vmatrix} n+a & 1 & 1 \cdots & 1 \\ n+a & 1+a & 1 \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n+a & 1 & \cdots & 1+a \end{vmatrix}$

$= (n+a)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a \end{vmatrix} = (n+a) \times \begin{vmatrix} 1+a & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1+a \end{vmatrix}$

$= (n+a)\begin{vmatrix} n-1+a & 1 & \cdots & 1 \\ \vdots & 1+a & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ n-1+a & 1 & \cdots & 1+a \end{vmatrix} = (n+a)(n-1+a)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a \end{vmatrix}$

$= (n+a)(n-1+a)(-1)\begin{vmatrix} 1+a & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1+a \end{vmatrix}_{(n-2)}$

$= \cdots = (-1)^{n-1}\prod_{i=1}^{n}(i+a)$.

5. $AV = VA$. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} V = V \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

(2) $V = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ $\begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 - a_3 & b_1 - b_3 & c_1 - c_3 \\ -a_2 & -b_2 & -c_2 \end{pmatrix} = \begin{pmatrix} b_1 & a_1 - c_1 & -b_1 \\ b_2 & a_2 - c_2 & -b_2 \\ b_3 & a_3 - c_3 & -b_3 \end{pmatrix}$

$a_2 = b_1$
$a_1 - a_3 = b_2$
$-a_2 = b_3$
$c_2 = -b_1$
$c_1 - c_3 = -b_2$
$-c_2 = -b_3$

$b_1 = -b_3$. $b_2 = a_1 - c_1 = a_1 - a_3$　$c_1 = a_3$

$b_1 - b_3 = a_2 - c_2 = 2a_2$　$c_2 = -a_2 = b_3$

$-b_2 = a_3 - c_3 = c_1 - c_3$　$2b_1$

$V = \begin{pmatrix} b_1 & b_2 & -b_1 \\ b_2 & 2b_1 & -b_2 \\ -b_1 & -b_2 & b_1 \end{pmatrix} \to \begin{pmatrix} b_1 & b_2 & 0 \\ b_2 & 2b_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

维数 2, 基 $(b_1, b_2, 0)$　$(b_2, 2b_1, 0)$.

(1) V 满足 $AV - VA = 0$.

对 $\forall V_1, V_2 \in F^n$, $\lambda \in R$,

$A(V_1 + V_2) = AV_1 + AV_2 \cdots$

6. $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$. $AB = (c_{ij})_{m\times p} = (\sum_{k=1}^{n} a_{ik} b_{kj})_{m\times p}$

充: $ABx = 0$ 的解为 $Bx = 0$ 的解,

$\begin{cases} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1p}x_n = 0 \\ c_{21}x_1 + c_{22}x_2 + \cdots + c_{2p}x_n = 0 \\ \vdots \\ c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mp}x_n = 0 \end{cases}$ $\begin{cases} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1p}x_n = 0 \\ \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{np}x_n = 0 \end{cases}$

必: $\operatorname{rank}(AB) = \operatorname{rank} B$, $AB = C_{m\times p}$.

$P_1 B Q_1 = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}$. $\operatorname{rank} B = r_1$

$P_3 C Q_3 = P_3 A B Q_3 = P_2 A Q_2 P_1 B Q_1$

$= P_2 A Q_2 \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}$