

中国科学技术大学 2016—2017 学年第二学期

线性代数 (B1) 期中考试

1. (4分 × 6 = 24分) 填空题.

(1) $\alpha_1 = (1, 3, 2)^T, \alpha_2 = (4, 4, 0)^T, \alpha_3 = (2, 5, 3)^T, \alpha_4 = (-1, 2, 3)^T$, 则 $\text{rank}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ 2 ✓

(2) $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix}$, 则 $A^{10} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix}$

(3) 设 A 为 n 阶方阵, $\det A = 5, A^*$ 为 A 的伴随方阵, 则 $\det A^* = 5^{n-1}$ ✓

(4) 设 $A = \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & 2 \\ 1 & 5 & 2 & 3 \end{pmatrix}, A_{ij}$ 为代数余子式, 则 $A_{14} - 3A_{24} + 2A_{34} - A_{44} =$ 6 ✓

(5) 若向量 $\beta = (3, 9, 6)$ 不能由向量组 $\alpha_1 = (1, 1, 2), \alpha_2 = (1, 2, -1), \alpha_3 = (1, -\lambda, 3)$ 线性表示, 则 $\lambda = -\frac{2}{3}$ ✓

(6) 设分块矩阵 $A = \begin{pmatrix} O & B \\ C & O \end{pmatrix}$, 其中 B, C 为 n 阶可逆方阵, O 为零方阵, 则 $(A^T)^{-1} = \begin{pmatrix} O & (B^T)^{-1} \\ (C^T)^{-1} & O \end{pmatrix}$ ✓

2. (5分 × 4 = 20分) 判断题 (判断下列命题是否正确, 并简要给出理由).

(1) 设 $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 5 & -2 \\ 5 & 11 & -4 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & -2 \\ 5 & 0 & 4 \\ 3 & 0 & 2 \end{pmatrix}$, 则 A 与 B 不相抵. \times $\text{rank } A = \text{rank } B = 2$

(2) 设数域 F 中的向量组 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性相关, $A \in F^{m \times 1}, (\beta_1, \beta_2, \dots, \beta_r) = (\alpha_1, \alpha_2, \dots, \alpha_m)A$, 则向量组 $\beta_1, \beta_2, \dots, \beta_r$ 也线性相关. \times $\beta_1 \neq 0, \beta_2 = 0, \dots, \beta_r = 0$

(3) A, B 为 n 阶方阵, 则 $\text{rank}(AB) = \text{rank}(BA)$. \times

(4) 设向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$ 的秩为 r , 且任何向量 $\alpha_i (1 \leq i \leq n)$ 均可以被 $\alpha_1, \alpha_2, \dots, \alpha_r$ 线性表示, 则 $\alpha_1, \alpha_2, \dots, \alpha_r$ 是 $\alpha_1, \alpha_2, \dots, \alpha_n$ 的一个极大线性无关组. \checkmark

3. (12分) 当 α 取何值时, $\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 2 \\ 3x_1 + 8x_2 - x_3 - 2x_4 = 0 \end{cases}$ 有解? 求出它的通解.

4. (16分) 设 n 阶方阵 $A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & -1 \\ -1 & -1 & -1 & \dots & -1 & -1 \end{pmatrix}$, 求 $\det A$ 及 A^{-1} .

5. (16分) 设 $\mathbb{P}_3[x]$ 为实数域 \mathbb{R} 上次数不超过 3 的多项式全体, 按多项式的加法数乘构成线性空间.

(1) 证明: $S = \{1, x+1, (x+1)^2, (x+1)^3\}$ 构成 $\mathbb{P}_3[x]$ 上的一组基;

(2) 求基 S 到自然基 $\{1, x, x^2, x^3\}$ 的过渡矩阵 T ;

(3) 求多项式 $5 + 7x - x^2 + 13x^3$ 在基 S 下的坐标.

6. 设方阵 $A = (a_{ij})_{n \times n}, e = \text{tr}(A) = \sum_{i=1}^n a_{ii}$, 已知 $\text{rank } A = 1$,

17:45

1. (1) $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 3 & 4 & 5 & 2 \\ 2 & 0 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & -8 & -1 & 5 \\ 0 & -8 & -1 & 5 \end{pmatrix}$
 $\rightarrow \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & -8 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{秩} = 2$

(2) $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \\ 2\alpha \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 4 & 2 \end{pmatrix}$

(3) $\det A = 5, \det A^* = (\det A)^{n-1} = 5^{n-1}$

(4) $A = \begin{pmatrix} 1 & 5 & 2 & -1 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & -1 & 2 \\ 1 & 5 & 2 & -1 \end{pmatrix} \quad 3 - (-3) = 6$

$A_{14} = - \begin{vmatrix} 0 & 3 & 1 \\ 0 & 0 & -1 \\ 1 & 5 & 2 \end{vmatrix} = - \begin{vmatrix} 0 & 3 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 3$

$A_{24} = \begin{vmatrix} 1 & 5 & 2 \\ 0 & 0 & -1 \\ 1 & 5 & 2 \end{vmatrix} = 0, \quad A_{34} = 0$

$A_{44} = \begin{vmatrix} 1 & 5 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} = -3$

(5) $\beta = k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3$
 $\begin{cases} 3 = k_1 + k_2 + k_3 \\ 9 = k_1 + 2k_2 - \lambda k_3 \\ b = 2k_1 - k_2 + 3k_3 \end{cases} \quad \text{增广}$

$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & -\lambda-1 & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -\lambda-\frac{2}{3} & b \end{pmatrix}$

$-\lambda - \frac{2}{3} = 0, \quad 0 \cdot k_3 = b \quad \text{增广} \Rightarrow \lambda = -\frac{2}{3}$

(6) $A = \begin{pmatrix} O & B \\ C & O \end{pmatrix} \quad A^T = \begin{pmatrix} O & C^T \\ B^T & O \end{pmatrix}$

$(A^T)^{-1} A^T = I, \quad (A^T)^{-1} \begin{pmatrix} O & C^T \\ B^T & O \end{pmatrix} = \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix}$

$\begin{pmatrix} M & N \\ P & Q \end{pmatrix} \begin{pmatrix} O & C^T \\ B^T & O \end{pmatrix} = \begin{pmatrix} NB^T & MC^T \\ QB^T & PC^T \end{pmatrix} = \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix}$

$NB^T = I_n, \quad (NB^T B^T)^{-1} = (B^T)^{-1}, \quad N = (B^T)^{-1}, \quad P = (C^T)^{-1}$

$\therefore (A^T)^{-1} = \begin{pmatrix} O & (B^T)^{-1} \\ (C^T)^{-1} & O \end{pmatrix}$

12:03

2. (1) $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 5 & -2 \\ 3 & 11 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -7 & 3 \\ 1 & 5 & -2 \\ 0 & -14 & b \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5 & -2 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$\text{rank } A = \text{rank } B = 2, \quad \text{不相抵} \quad \times$

(2) $(\beta_1, \beta_2, \dots, \beta_r) = (\alpha_1, \alpha_2, \dots, \alpha_m) \begin{pmatrix} a_{11} & \dots & a_{1r} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mr} \end{pmatrix} \quad \text{取 } A = (1, 1, \dots, 1)$

$\beta_i = a_{i1} \alpha_1 + a_{i2} \alpha_2 + \dots + a_{im} \alpha_m$

α 相抵, $k_1 \alpha_1 + \dots + k_m \alpha_m = 0 \quad \forall$ 那那为 0

又 $k_1 \beta_1 + \dots + k_r \beta_r = 0$

$= a_{11} \alpha_1 + a_{12} \alpha_2 + \dots + a_{1m} \alpha_m + \dots + a_{r1} \alpha_1 + a_{r2} \alpha_2 + \dots + a_{rm} \alpha_m$

$= (a_{11} + \dots + a_{r1}) \alpha_1 + (a_{12} + \dots + a_{r2}) \alpha_2 + \dots + (a_{1m} + \dots + a_{rm}) \alpha_m$

$= \sum_{i=1}^m a_{ij} \alpha_j + \dots + \sum_{i=1}^r a_{mi} \alpha_m$

即 $\sum_{i=1}^r a_{ij} \alpha_j + \dots + \sum_{i=1}^r a_{mi} \alpha_m = 0$

又 $k_j \alpha_j = \sum_{i=1}^m a_{ji} \alpha_i$, 可得 k_j 不全为 0

\therefore 线性无关. \checkmark

(3) $A = (a_{ij})_{n \times n}, \quad B = (b_{ij})_{n \times n}, \quad C = AB = \left(\sum_{k=1}^n a_{ik} b_{kj} \right)_{n \times n}$

$D = BA = \left(\sum_{k=1}^n b_{ik} a_{kj} \right)_{n \times n}$

$\text{rank}(AB), \text{rank}(BA)$ 不一定相等. \times

(4) 取 $\alpha_1, \dots, \alpha_n$ 线性无关 \rightarrow 线性无关. \checkmark

线性相关 \rightarrow 不可表示 α_i . \checkmark

12:26

3. $\begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 3 & 8 & -1 & -2 & 0 \\ 2 & 5 & -2 & 1 & a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & 2 & 8 & -14 & -6 \\ 0 & 1 & 4 & -7 & a-4 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 & 2 \\ 0 & 2 & 8 & -14 & -6 \\ 0 & 0 & 0 & 0 & a-1 \end{pmatrix} \quad \text{秩} = 3 \Rightarrow a = 1$

$\begin{cases} 7x_1 + 2x_2 - 3x_3 + 4x_4 = 2 \\ 2x_2 + 8x_3 - 14x_4 = -6 \end{cases}$

取 $x_3 = t_1, \quad x_4 = t_2, \quad x_2 = \frac{-6 + 14t_2 - 8t_1}{2}$

$x_1 = \frac{2 - 4t_2 + 3t_1 - (-6 + 14t_2 - 8t_1)}{7}$

$= \frac{8 - 18t_2 + 11t_1}{7}$

\therefore 通解为 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{8}{7} \\ -3 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} -\frac{18}{7} \\ 7 \\ 0 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$(\eta_1, \eta_2, \eta_3, \eta_4) = (12t_1 - 18t_2 + 7, -9t_1 + 7t_2 - 3, t_1, t_2)$

12:34

4. $\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & -1 & -1 \\ -1 & -1 & \dots & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -2 & 0 \end{vmatrix}$

$= 1 \times (-1)^{n+1} \times (-2) \times \dots \times (-2) = (-1)^{n+1} (-2)^{n-1} = (-1)^{n+1} (-1)^{n-1} 2^{n-1} = 2^{n-1}$ ✓

$AA^* = \det(A)I, \quad A^{-1} = \frac{A^*}{\det A}$

$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & -2 & -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1 & \dots & 0 & 1 & -\frac{1}{2} & \dots & -\frac{1}{2} & 0 \\ \frac{1}{2} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & \dots & -\frac{1}{2} & 0 \\ \frac{1}{2} & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 2 & 0 & \dots & 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 0 & 0 & 0 & 1 & \dots & -1 & -1 \\ -1 & -1 & \dots & -1 & -1 & 0 & 0 & 0 & \dots & -1 & -1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 2 & 2 & \dots & 2 & 1 & -1 & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & -\frac{1}{2} & \dots & -\frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$

8:20

5. (1) $\mathbb{P}_3[x], \deg p_{ij} = b, \dim V = 4$. \checkmark

S 中有 4 个元素, 且互不相同, 线性无关.

$\therefore S$ 中的元素为极大无关组, 构成一组基.

(2) $(1-x)(1+x)(1+x^2) = (1-x^2)(1+x^2) = 1-x^4$

令 $1-x = m$, 则 $x = 1-m$

$(1-m)(1-m^2)(1-m^4) = (1-m)(1-m^2)(1-m^4)$

$\therefore T = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(3) $5 + 7x - x^2 + 13x^3 = (1-x)(1+x)(1+x^2) \begin{pmatrix} 5 \\ 7 \\ -1 \\ 13 \end{pmatrix}$

$= (1-x)(1+x)(1+x^2) \begin{pmatrix} 5 \\ 7 \\ -1 \\ 13 \end{pmatrix}$

\therefore 坐标为 $\begin{pmatrix} -16 \\ 48 \\ -10 \\ 13 \end{pmatrix}$

6. (1) $\text{rank}(A) = 1, A \neq O$ 且各行 (或列) 线性相关.

设各行为 α_i , 各列为 β_j . $\text{行} \alpha_i = \alpha \beta$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \beta = (\beta_1, \beta_2, \dots, \beta_n)$

$A = (\alpha_i \beta_j)_{n \times n} = \left(\sum_{k=1}^n \alpha_{ik} \beta_{kj} \right)_{n \times n}$

即 $\forall b_{ij}, b_{ij} = \alpha_{i1} \beta_{1j} + \alpha_{i2} \beta_{2j} + \dots + \alpha_{in} \beta_{nj}$

$= \alpha_i (\beta_1 \beta_2 \dots \beta_n) = \alpha_i \beta$

$C = (c_{ij})_{n \times n} = (\alpha_{ij} \sum_{k=1}^n \beta_{kj})_{n \times n}$

即 $\forall c_{ij}, c_{ij} = \alpha_{ij} (\beta_1 + \beta_2 + \dots + \beta_n)$

$A^T = (\alpha^T \beta) (\alpha^T \beta) = \alpha^T \beta \alpha^T \beta = C^T$