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2021年12月2日 星期四　　下午9:05

中国科学技术大学 2015—2016 学年第二学期
线性代数（B1）期中考试

1. (4 分 ×5 = 20 分) 填空题.

(1) 设矩阵 $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$, 则 $\operatorname{rank} A^TA =$ 2 ✓.

(2) 以 $\mathbb{R}^3$ 中三个向量 $e_1 = (2,1,0)$, $e_2 = (6,3,2)$, $e_3 = (1,0,5)$ 为基, 向量 $(2,-1,2)$ 的坐标是 $(26,-9,4)^T$ ✓.

(3) 矩阵 $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ 的逆矩阵 A^{-1} 的伴随矩阵是 $\begin{pmatrix} -2 & -2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -6 \end{pmatrix}$ $\frac{1}{2}A$

(4) 设 $\mathbb{R}^4$ 中向量组 $\alpha_1 = (1,2,-1,1)$, $\alpha_2 = (2,0,t,0)$, $\alpha_3 = (0,-4,5,-2)$ 线性相关, 则 $t =$ 3 ✓.

(5) 设矩阵 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $n \in \mathbb{N}^*$, 则 $A^n =$ $\begin{pmatrix} 1 & n & -\frac{n(n-1)}{2} \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix}$ ✓

2. (5 分 ×4 = 20 分) 判断题.

(1) 若非齐次线性方程组 $Ax = b$ 解唯一, 则对应的齐次线性方程组 $Ax = 0$ 只有零解.

(2) 若矩阵 A, B 满足 AB, BA 都有定义, 则 $\det AB = \det BA$.

(3) 若向量 β 不能由向量组 $\alpha_1, \cdots, \alpha_s$ 线性表示, 则向量组 $\beta, \alpha_1, \cdots, \alpha_s$ 线性无关.

(4) 设 W 是 $\mathbb{R}^{n\times n}$ 上所有行列式为 0 的矩阵全体, 则 W 是 $\mathbb{R}^{n\times n}$ 的子空间.

3. (14 分) 已知线性方程组 $\begin{cases} \lambda x_1 + x_2 + x_3 = \lambda - 3 \\ x_1 + \lambda x_2 + x_3 = -2 \\ x_1 + x_2 + \lambda x_3 = -2 \end{cases}$.

(1) 讨论 λ 取何值时, 方程组无解、有唯一解和有无穷多组解.

(2) 当方程组有无穷多组解时, 求出通解.

4. (20 分) 设矩阵 $A = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n+1 \end{pmatrix}$.

(1) 计算矩阵 A 的行列式;　(2) 计算矩阵 A 的逆矩阵.

5. (14 分) 给定 I: $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$, $A_3 = \begin{pmatrix} -6 & 4 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix}$.

(1) 证明向量组 (I) 是线性空间 $\mathbb{R}^{2\times 2}$ 的一组基.

(2) 给定 II: $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, 求 (I) 到 (II) 的过渡矩阵.

(3) 求 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 在基 (I) 下的坐标.

6. (12 分) 设 F 为数域, $A \in F^{n\times n}$, $A^2 = I$, I 是 n 阶单位阵.

(1) 证明: $\operatorname{rank}(A+I) + \operatorname{rank}(A-I) = n$.

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1. (1) $A^TA = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 8 & 11 & 14 \\ 8 & 13 & 18 & 23 \\ 11 & 18 & 25 & 32 \\ 14 & 23 & 32 & 41 \end{pmatrix}$

$\operatorname{rank}(A^TA) = \operatorname{rank}\begin{pmatrix} 5 & 8 & 11 & 14 \\ 8 & 13 & 18 & 23 \\ 11 & 18 & 25 & 32 \\ 14 & 23 & 32 & 41 \end{pmatrix}$

$\det(A^TA) = \det A^T \det A = 0.$　$\operatorname{rank} = 2.$

(2) $(e_1\ e_2\ e_3) = \begin{pmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 5 \end{pmatrix}$　$\alpha = (2,-1,2)$. $= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$

$\begin{cases} 2\lambda_1 + 6\lambda_2 + \lambda_3 = 2 \\ \lambda_1 + 3\lambda_2 = -1 \\ 2\lambda_2 + 5\lambda_3 = 2 \end{cases}$　$\begin{pmatrix} 2 & 6 & 1 & 2 \\ 1 & 3 & 0 & -1 \\ 0 & 2 & 5 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 6 & 1 & 2 \\ 0 & 0 & -1 & ~~-4~~ \\ 0 & 2 & 5 & 2 \end{pmatrix}$

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$\to \begin{pmatrix} 2 & 6 & 1 & 2 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & -1 & -4 \end{pmatrix}$　$\lambda_3 = 4$. $2\lambda_2 + 5\lambda_3 = 2$, $\lambda_2 = -9$

$2\lambda_1 + 6\lambda_2 + \lambda_3 = 2$. $\lambda_1 = 26$.

$2\lambda_1 = 2 - 4 + 6\times 9 = -2 + 54 = 52$

∴ 坐标 (26, -9, 4).

(3) $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -\frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.　$AA^* = \det(A)I$

$A^{-1} = \frac{A^*}{\det A}$

$\det A^{-1} = -\frac{5}{4} - \frac{1}{4} - \frac{1}{2} = -2$　$A^{-1}(A^{-1})^* = \det(A^{-1})I$.

$(A^{-1})^* = \det(A^{-1})A$.

$(A^{-1})^* = -2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} ~~-\frac{3}{2}~~ & ~~-\frac{3}{2}~~ & ~~-\frac{3}{2}~~ \\ ~~-\frac{3}{2}~~ & ~~-3~~ & ~~-\frac{3}{2}~~ \\ ~~-\frac{3}{2}~~ & ~~-\frac{3}{2}~~ & ~~-\frac{9}{2}~~ \end{pmatrix}$

$A^{-1} = \frac{A^*}{\det A}$.

$A^* = \begin{pmatrix} 5 & -2 & -1 \\ -2 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$　$= \begin{pmatrix} -2 & -2 & -2 \\ -2 & -4 & -2 \\ -2 & -2 & -6 \end{pmatrix}$

(4) $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & -4 & 0 \\ -1 & t & 5 & 0 \\ ~~1~~ & ~~0~~ & ~~-2~~ & ~~0~~ \end{pmatrix}$　$\begin{cases} \lambda_1 + 2\lambda_2 = 0 \\ 2\lambda_1 - 4\lambda_3 = 0 \\ -\lambda_1 + t\lambda_2 + 5\lambda_3 = 0 \\ ~~\lambda_1 - 2\lambda_3 = 0~~ \end{cases}$　有非零解.

$\to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & t+2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3-t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$　$t \neq -2$　$t = 3$

(5) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = I + B.$

$A^n = (I+B)^n = \sum_{i=0}^{n} C_n^i I^{n-i} B^i = \sum_{i=0}^{n} C_n^i B^i.$

$B^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$B^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$A^n = C_n^0 B^0 + C_n^1 B + C_n^2 B^2$

$= I + \begin{pmatrix} 0 & n & 0 \\ 0 & 0 & -n \\ 0 & 0 & 0 \end{pmatrix} + \frac{n(n-1)}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & n & -\frac{n(n-1)}{2} \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix}$

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2. (1) $Ax=b$ 有解 ⇒ $\operatorname{rank} A = \operatorname{rank}(A, b) = \operatorname{rank}(A\ 0) = n$ 唯一 ✓

(2) ✓ $\det(A)\det(B)$ ✗ 反例. 如 0.

(3) $\alpha_1 - \alpha_n$ 可有关. ✗

(4) $\det(W) = 0$. $\det(W_1 + W_2)$ ✗

W不可逆　$W_1 + W_2$ 不可逆?

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3. (1) $\begin{pmatrix} \lambda & 1 & 1 & \lambda-3 \\ 1 & \lambda & 1 & -2 \\ 1 & 1 & \lambda & -2 \end{pmatrix} = (A, b).$

$\det(A) = \lambda^3 + 2 - 3\lambda = (\lambda-1)(\lambda^2+\lambda-2)$
$= (\lambda-1)(\lambda-1)(\lambda+2)$
$= (\lambda-1)^2(\lambda+2)$

$$\begin{array}{r} \lambda^2 + \lambda - 2 \\ \lambda - 1 \,\big)\, \overline{\lambda^3 + 0\lambda^2 - 3\lambda + 2} \\ \underline{\lambda^3 - \lambda^2} \\ \lambda^2 - 3\lambda \\ \underline{\lambda^2 - \lambda} \\ -2\lambda + 2 \\ \underline{-2\lambda + 2} \\ 0 \end{array}$$

$\begin{pmatrix} -2 & 1 & 1 & -5 \\ 1 & -2 & 1 & -2 \\ 1 & 1 & -2 & -2 \end{pmatrix}$　$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$　$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

$\lambda = 1$, $\det A = 0$, $\operatorname{rank} A = \operatorname{rank}(A, b) = 1$.

有解. 有无穷多组解.

$\lambda = -2$, $\det A = 0$, $\operatorname{rank} A = 2$　$\begin{pmatrix} 0 & 0 & 0 & -9 \\ 1 & -2 & 1 & -2 \\ 1 & 1 & -2 & -2 \end{pmatrix}$

无解.

$\lambda = 3$, $\operatorname{rank} A = 3$,　$\begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & -2 \\ 1 & 1 & 3 & -2 \end{pmatrix}$　$\begin{pmatrix} 3 & 1 & 1 & 0 \\ -2 & 2 & 0 & -2 \\ -8 & -2 & 0 & -2 \end{pmatrix}$

有一组解.　$\to \begin{pmatrix} 3 & 1 & 1 & 0 \\ -2 & 2 & 0 & -2 \\ -10 & 0 & 0 & -4 \end{pmatrix}$

$\lambda \neq 1$. $\lambda \neq -2$. $\lambda \neq 3$. 有一组解.

(2) $\lambda=1$, $\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -2 \end{pmatrix}$ 无穷多解.

$x_1+x_2+x_3=-2$. 取 $x_2=t_1$, $x_3=t_2$, 则 $x_1=-2-t_1-t_2$

通解 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$.

4. (1) $\begin{vmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n+1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -3 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & n-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1-n & n \end{vmatrix} = \Delta_n$

$= (-1)^{n+1} \begin{vmatrix} -1 & 2 & & & \\ & -2 & 3 & & \\ & & -3 & \ddots & \\ & & & \ddots & n-1 \\ & & & & 1-n \end{vmatrix} + n \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 \\ -1 & 2 & 0 & \cdots & 0 \\ 0 & -2 & 3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \end{vmatrix}$

$= (-1)^{n+1}(-1)(-2)\cdots(-n+1) + n\Delta_{n-1}$

$= (-1)^{n-1}(-1)^{n-1}(1\times 2\times\cdots(n-1)) + n\Delta_{n-1}$

$= \prod_{i=1}^{n-1} i + n\Delta_{n-1}$.

$n=2$, $\Delta_1=2$.

$\therefore \Delta_n = \prod_{i=1}^{n-1} i + n\Delta_{n-1} = \prod_{i=1}^{n-1} i + n\left(\prod_{i=1}^{n-2} i + (n-1)\Delta_{n-2}\right)$

$= \cdots = \prod_{i=1}^{n-1} i + n\prod_{i=1}^{n-2} i + n(n-1)\prod_{i=1}^{n-3} i + \cdots + n! + n!\Delta_1$

$= \sum_{j=1}^{n-1} \frac{n!}{(j+1)} + \cancel{2}n!$ $n!$

(2) $\det A \neq 0$, A 可逆.

$A_{ii} = \frac{1}{i}\left(1+\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{i}\right)n!$

$A_{ij} = (-1)\frac{1}{ij}n!$

$(A\ I) = \left(\begin{array}{cccccc|cccccc} 2 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 1 & \cdots & 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 4 & 1 & \cdots & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 5 & \cdots & 1 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right)$

$= \left(\begin{array}{ccccccc|cccccc} 2 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 3 & 0 & \cdots & 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -3 & 4 & \cdots & 0 & 0 & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & n-1 & 0 & & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-n & n & 0 & 0 & 0 & \cdots & -1 & 1 \end{array}\right)$

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5. (1) $\mathbb{R}^{2\times 2}$ 维数为 4.

A_1, A_2, A_3, A_4 线性无关.

极大无关组, 个数为 4 的一组基.

→ 证 $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ 可逆.

6. (1) $\operatorname{rank}(A+I) + \operatorname{rank}(A-I) = \operatorname{rank}\begin{pmatrix} A+I & 0 \\ 0 & A-I \end{pmatrix}$

$\begin{pmatrix} A+I & 0 \\ 0 & A-I \end{pmatrix} \to \begin{pmatrix} A+I & 0 \\ -A & A-I \end{pmatrix} \to \begin{pmatrix} I & A-I \\ -A & A-I \end{pmatrix}$

$\to \begin{pmatrix} I-A^2 & (A-I)A+A-I \\ -A & A-I \end{pmatrix} = \begin{pmatrix} 0 & A^2-A+A-I \\ -A & A-I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A-I \end{pmatrix}$

$A \neq I$, $\operatorname{rank}\begin{vmatrix} 0 & 0 \\ 0 & A-I \end{vmatrix} = n$.

(2) 对 $\forall x_1, x_2 \in W_1$. $Ax_1 = x_1$. $Ax_2 = x_2$.

$A(x_1+x_2) = Ax_1 + Ax_2 = x_1 + x_2$

$A(\lambda$

$\left.\begin{array}{l} \dim W_1 = n - \operatorname{rank}(A-I) \\ \dim W_2 = n - \operatorname{rank}(A+I) \end{array}\right\} = n = \dim F^n$.

W_1 基为 $\alpha_1 \cdots \alpha_r$. W_2 基为 $\beta_1 - \beta_{n-r}$

证 $\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_{n-r}$ 线性无关.