20～21期末

2021年12月27日 星期一　　下午9:46

中国科学技术大学 2020—2021 学年第一学期
线性代数（B1）期末考试

1. (5 分 ×5 = 25 分) 填空题.

(1) 方阵 $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ 的特征值是 $\sqrt{3}$, $-\sqrt{3}$

(2) 3 阶实对称矩阵组成的集合恰有 ____ 个相合等价类.

(3) 实二次型 $Q(x_1, x_2, x_3, x_4) = \sum_{1 \leqslant i < j \leqslant 4} (x_i - x_j)^2$ 的正惯性指数等于 ____.

(4) 设 $\mathbb{R}^3$ 中的线性变换 $\mathscr{A}$ 满足 $\mathscr{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$, 其中 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ 是 $\mathbb{R}^3$ 中任意的向量, 则 $\mathscr{A}$ 在自然基下的矩阵是 ____.

(5) 设 $\mathscr{A}$ 是 n 维欧式空间 V 上的线性变换: $\mathscr{A}(\alpha) = \alpha - 2(\alpha, \gamma)\gamma$, 其中 γ 是 V 中给定的单位向量, 则 $\mathscr{A}$ 的 n 个特征值为 ____.

将 γ 作为标准正交基中的一个

$\mathscr{A}(\gamma) = \gamma - 2(\gamma, \gamma)\gamma = \gamma - 2\gamma = -\gamma \Rightarrow -1.$

$\mathscr{A}(e_2) = e_2 - 2(e_2, \gamma)\gamma = e_2 \Rightarrow 1.$

2. (5 分 ×5 = 25 分) 判断题.

(1) n 维线性空间 V 中同一个线性变换在两组不同的基本下的矩阵彼此相合.

(2) 任何一个 n 阶实方阵都实相似于上三角矩阵.　书：任何一个复方阵都可以..

(3) 每一个正交矩阵都正交相似于对角矩阵.

$AA^T = I$ 正交.　　要求有 n 个 λ，是实数解

$PP^T = I$, $P^{-1} = P^T$.　$P^{-1}AP = \begin{pmatrix} \ddots \end{pmatrix}^{\lambda}$

(4) 设 A, B 都是 n 阶实方阵, 若 A 可逆, 则 AB 与 BA 相似.

(5) 设 A 是 n 阶实对称方阵, 若 A 的每一个顺序主子式都是非负的, 则 A 半正定.

3. (12 分) 设 $\mathbb{R}^3$ 的线性变换 $\mathscr{A}$ 将 $\alpha_1 = (2,3,5)^T$, $\alpha_2 = (0,1,2)^T$, $\alpha_3 = (1,0,0)^T$ 变换为 $\beta_1 = (1,2,0)^T$, $\beta_2 = (2,4,-1)^T$, $\beta_3 = (3,0,5)^T$.

(1) 求 $\mathscr{A}$ 在基 $\beta_1, \beta_2, \beta_3$ 下的矩阵; (2) 求 $\mathscr{A}$ 在自然基下的矩阵.

4. (16 分) 设 V 是 3 维欧式空间, 由基 $\alpha_1, \alpha_2, \alpha_3$ 给出的度量矩阵 $= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 10 & -2 \\ 1 & -2 & 2 \end{pmatrix}$. 请由 $\alpha_1, \alpha_2, \alpha_3$ 按现在的顺序进行 Schmidt 正交化给出一组标准正交基.

5. (12 分) 给定二次曲面在直角坐标系下的方程是 $2x^2 + 6y^2 + 2z^2 + 8xz = 1$. 将它通过正交变换化为标准方程, 并指出这曲面的类型.

6. (10 分) 设 A, B 是两个 n 阶实对称矩阵, 满足 $AB = BA$. 求证: 存在 n 阶正交方阵 P, 使得 P^TAP 与 P^TBP 都是对角矩阵.

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1. (1) $\det(\lambda I - A) = \det\begin{pmatrix} \lambda & 0 & -1 \\ 0 & \lambda-2 & 0 \\ -3 & 0 & \lambda \end{pmatrix}$　　$\lambda^2(\lambda-2) - 3(\lambda-2) = 0$

$(\lambda^2-3)(\lambda-2) = 0$

$= -2\lambda^2 + 6 = -2(\lambda^2 - 3) = -2(\lambda-\sqrt{3})(\lambda+\sqrt{3}) = 0.$

$\lambda = \sqrt{3}$　$\begin{pmatrix} \sqrt{3} & 0 & -1 \\ 0 & -2 & 0 \\ -3 & 0 & \sqrt{3} \end{pmatrix}$, $x_1 = 1$, $x_2 = 0$, $x_3 = \sqrt{3}$

$c_1(1, 0, \sqrt{3})$.

$\lambda = -\sqrt{3}$　$\begin{pmatrix} -\sqrt{3} & & \\ & & -\sqrt{3} \end{pmatrix}$　$x_1 = 1$　$x_3 = -\sqrt{3}$　　$\lambda = 2$.

$c_2(1, 0, -\sqrt{3})$.

(2) $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$　　~~2~~ 10 错误

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$

$\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} -1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$

$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

(3) $Q = (x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_4)^2$
$+ (x_1-x_3)^2 + (x_1-x_4)^2 + (x_2-x_4)^2$

$y_1 = x_1 - x_2$, $y_2 = x_1 - x_3$. $y_3 = x_1 - x_4$.

$= y_1^2 + (y_1-y_2)^2 + (y_2-y_3)^2 + y_3^2 + (y_1-y_3)^2$

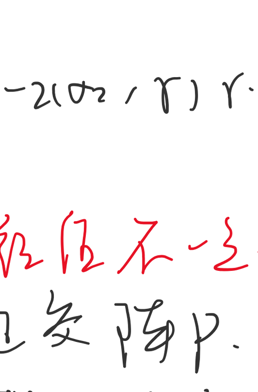
$= 3y_1^2 + 3y_3^2 + 2y_2^2 - 2y_1y_2 - 2y_2y_3 - 2y_1y_3$

$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2+\frac{1}{3} & -1 \\ 0 & -1 & 3+\frac{1}{3} \\ 1 & & \\ 0 & & \\ 0 & & \end{pmatrix}$　$\frac{10}{3}$　$\frac{10}{3}$　$-\frac{3}{10}$　　3.

(4) $\mathscr{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$　　$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathscr{A}e_1 = \mathscr{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$　$\mathscr{A}e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$　$\mathscr{A}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(5) $\mathscr{A}(\alpha) = \alpha - 2(\alpha, \gamma)\gamma$.



设记 $\beta = \alpha - (\alpha, \gamma)\gamma$.

正交变换. 特征值 ± 1. $A^TA = I$

取基 $\alpha_1, \cdots, \alpha_n$.　　$-1, 1$ $(n-1)$ 重)

$\mathscr{A}\alpha_1 = \alpha_1 - 2(\alpha_1, \gamma)\gamma$.　$A_2 = \alpha_2 - 2(\alpha_2, \gamma)\gamma$.

2. (1) × ~~相合~~

(2) √ ~~×~~ 定理.　实方阵的特征值不一定是实数

(3). 正交 $A^TA = I$.　正交相似. 正交阵 P. $P^TP = I$,

使 $P^{-1}AP = P^TAP$.　正交阵相似(合)于对~~角~~ √.　不是实数.

(4). A 可逆.　$A^{-1}(AB)A = BA$. √

(5) 也有可能正定? (全正)

含0.?　×　　如 $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

3. $\mathscr{A}(\alpha_1\ \alpha_2\ \alpha_3) = (\beta_1\ \beta_2\ \beta_3) = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix}$

(1) $\mathscr{A}(\beta_1\ \beta_2\ \beta_3) = \mathscr{A}((\alpha_1\ \alpha_2\ \alpha_3)A)$
$= \mathscr{A}(\alpha_1\ \alpha_2\ \alpha_3)A$
$= (\alpha_1\ \alpha_2\ \alpha_3)A \cdot A$
$= (\beta_1\ \beta_2\ \beta_3)A^{-1}AA$
$= (\beta_1\ \beta_2\ \beta_3)A$

$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ 0 & -5 & 3 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 & 1 & -4 & 2 \\ 0 & 1 & 0 & 0 & -5 & 3 \\ -1 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & -5 & 3 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & -5 & 3 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 9 & -5 \\ -10 & -23 & 15 \\ -7 & -16 & 13 \end{pmatrix}$

(2) $\mathscr{A}(e_1\ e_2\ e_3) = \mathscr{A}((\alpha_1\ \alpha_2\ \alpha_3)(\alpha_1\ \alpha_2\ \alpha_3)^{-1})$
$= \mathscr{A}(\alpha_1\ \alpha_2\ \alpha_3)(\alpha_1\ \alpha_2\ \alpha_3)^{-1}$
$= (\beta_1\ \beta_2\ \beta_3)\ (\alpha_1\ \alpha_2\ \alpha_3)^{-1}$
$= (e_1\ e_2\ e_3)\underline{(\beta_1\ \beta_2\ \beta_3)(\alpha_1\ \alpha_2\ \alpha_3)^{-1}}$.

$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix}\begin{pmatrix} 0 & 2 & -1 \\ 0 & -5 & 3 \\ 1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -20 & 11 \\ 0 & -16 & 10 \\ 5 & -13 & 7 \end{pmatrix}$

4. $(\alpha_1, \alpha_1) = 1$, $|\alpha_1| = 1$. $e_1 = \alpha_1$.

$(\alpha_1, \alpha_2) = 0$, α_1, α_2 已正交.

$(\alpha_2, \alpha_2) = 10$, $|\alpha_2| = \sqrt{10}$. $e_2 = \frac{\alpha_2}{\sqrt{10}}$.　　$\frac{-2}{\sqrt{10}} \times \frac{1}{\sqrt{10}} = \frac{1}{5}$

$\beta_3 = \alpha_3 - (\alpha_3, e_1)e_1 - (\alpha_3, e_2)e_2$
$= \alpha_3 - (\alpha_3, \alpha_1)\alpha_1 - (\alpha_3, \frac{\alpha_2}{\sqrt{10}})\frac{\alpha_2}{\sqrt{10}}$
$= \alpha_3 - \alpha_1 + \frac{1}{5}\alpha_2$.

$(\beta_3, \beta_3) = (\alpha_3, \alpha_3 - \alpha_1 + \frac{1}{5}\alpha_2) - (\alpha_1, \alpha_3 - \alpha_1 + \frac{1}{5}\alpha_2) + (\frac{1}{5}\alpha_2, \alpha_3 - \alpha_1 + \frac{1}{5}\alpha_2)$　$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & & \end{pmatrix}$

$= 2 - 1 - \frac{2}{5} - 1 + 1 - \frac{2}{5} + \frac{10}{25} = \frac{3}{5}$.

$\therefore e_3 = \frac{\beta_3}{\sqrt{\frac{3}{5}}} = \sqrt{\frac{5}{3}}(\alpha_3 - \alpha_1 + \frac{1}{5}\alpha_2)$.

正交基为 α_1, $\frac{\alpha_2}{\sqrt{10}}$, $\sqrt{\frac{5}{3}}(\alpha_3 - \alpha_1 + \frac{1}{5}\alpha_2)$.

5.

$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{\sqrt{2}}{2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \\ \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \\ \frac{1}{2} & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

$= \tilde{x}^2 + 3\tilde{y}^2 - 3\tilde{z}^2 = 1$.　单叶双曲面.

6. A 为实对称阵, 则存在正交阵 P_1, 使得

$P_1^TAP_1 = P_1^{-1}AP_1 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

$AB = BA$, 也是实对称阵.

存在正交阵 P_2 使 $P_2^TABP_2 = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$

$AB = BA$,

$\therefore (P_1^TAP_1)(P_1^TBP_1) = (P_1^TBP_1)(P_1^TAP_1)$

其中 $P_1^TBP_1 = (B_1\ B_2\ \cdots\ B_n)$.

$\therefore$ 存在 n_i 阶正交阵 T_i, 使 $T_i^TB_iT_i$ 为对角阵.

取 $P_2 = (T_1\ \cdots\ T_n)$, $P = P_1P_2$

P_2, P 都是正交阵.

$P^TBP = \mathrm{diag}(T_i^TB_iT_i)$..