

16~17期末

2022年1月8日 星期六 下午3:54

中国科学技术大学 2016—2017 学年第一学期 线性代数 (B1) 期末考试

1. (5分 × 5 = 20分) 填空题.
- (1) 若 $\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 1 \\ x_1 - x_2 + x_3 + x_4 = 2 \end{cases}$ 有解, 则参数 $t = \underline{5}$.
 - (2) 若向量 $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 2, 3), \alpha_3 = (2, 3, t)$ 生成的 \mathbb{R}^3 中的 2 维子空间, 则参数 $t = \underline{4}$.
 - (3) 设 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, 则 A 的相合规范型为 $\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$.
 - (4) 二次曲面方程 $2x^2 - 3y^2 - 3z^2 - 2yz - 5 = 0$ 表示的曲面类型是 双叶双曲面.
 - (5) 实二次型 $Q(x, y, z) = x^2 + y^2 + 3z^2 + 2xy + 2xz + 2yz$ 为正定且仅当参数 t 满足 $\underline{1 < t < 4}$.
2. (5分 × 5 = 25分) 判断题.
- (1) 设 A 是 n 阶方阵, 则 $\text{rank } A = \text{rank } A^T$. 对
 - (2) 若 0 是矩阵 A 的特征值, 则 A 一定是奇异矩阵. 对
 - (3) 设 A 是 n 阶方阵, 若对任意 n 维列向量 x 都有 $x^T Ax = 0$, 则 A 为反对称方阵. 对
 - (4) 若 A, B 是 n 阶正定矩阵, 则 AB 也是 n 阶正定矩阵. 错
 - (5) 设 A 是 n 阶方阵, 则 A 是正交矩阵且仅当 A 的 n 个行向量组成的 n 维实数空间的标准正交基. 对
3. (14分) 在 \mathbb{R}^3 中定义线性变换 $\mathcal{A}(x, y, z) = (x + 2y, x - 3z, 2y - z)$.
- (1) 求 \mathcal{A} 在基 $\alpha_1 = (1, 0, 0), \alpha_2 = (1, 1, 0), \alpha_3 = (1, 1, 1)$ 下的表示矩阵.
 - (2) 是否存在基 $\beta_1, \beta_2, \beta_3$, 使得 \mathcal{A} 在该基下的表示矩阵为 $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$.
4. (14分) 设 $V = \{f(x) = a_0 + a_1x + a_2x^2\}$ 为次数不超过 2 的实系数多项式构成的线性空间.
- (1) 证明: $f(x), g(x) = f(0)g(0) + f(1)g(1) + f(2)g(2)$ 定义了 V 上的一个内积.
 - (2) 应用 Schmidt 正交化方法将向量组 $\{1, x\}$ 改造成相对 (1) 中所定义内积的标准正交向量组.
5. (14分) 设 M 是 $2n$ 阶方阵 $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$, 其中 A 是满足 $A^2 = I$ 的 n 阶对称方阵.
- (1) 求矩阵 M 的所有特征值.
 - (2) 可逆矩阵 P , 使得 $P^{-1}MP$ 为对角矩阵.
6. (8分) 假设 A 和 B 都是 n 阶正定矩阵, 证明: $\det A \cdot \det B \leq \left(\frac{1}{n} \text{tr } AB\right)^n$.

1. (1) $\begin{pmatrix} 1 & -2 & 1 & -1 & | & 1 \\ 1 & -1 & 1 & 1 & | & 2 \\ 3 & -6 & 3 & 1 & | & t \end{pmatrix} \quad t = 5$

$\rightarrow \begin{pmatrix} 1 & -2 & 1 & -1 & | & 1 \\ 0 & 1 & 0 & 2 & | & 1 \\ 0 & 2 & 0 & 4 & | & t-3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & -1 & | & 1 \\ 0 & 1 & 0 & 2 & | & 1 \\ 0 & 0 & 0 & 0 & | & t-5 \end{pmatrix}$

1. (2) $17 = 2 \cdot 1 + 2 \cdot 3 \quad t = 4$

$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & -3 & | & -\frac{1}{3} \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$-1 \times -\frac{1}{3} = \frac{1}{3} \quad -\frac{1}{3} + \frac{1}{3}$

1. (3) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

$= 2x^2 - 3y^2 - \frac{3}{2}z^2 - 5 = 0$ 为双叶双曲面.

1. (5) $Q = x^2 + y^2 + 3z^2 + 2xy + 2xz + 2yz$

$\begin{pmatrix} 1 & t & t \\ t & 1 & 1 \\ t & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & 1-t \\ 0 & 1-t & 3-t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & 0 \\ 0 & 1-t & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$t \leq 1-t^2 > 0 \quad t \in (-1, 1) \quad -1 < t < 1$

2. (1) $\text{rank } A = r \rightarrow P \Lambda Q = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, A = P^T \Lambda Q^T$

$\text{rank } A \cdot A \rightarrow \text{rank } P^T \Lambda Q^T P^T \Lambda Q^T = r$

$\rightarrow 0$ 是 A 的特征值. $(0-A)x = 0$ 有非零解.

$-Ax = 0$ 有非零解. $Ax = 0$ 有非零解. 对

1. (3) $(\lambda - \lambda_n) A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\lambda_1 \dots \lambda_n) \begin{pmatrix} a_{11} - \lambda_{11} & & \\ & \ddots & \\ & & a_{nn} - \lambda_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= \left(\sum_{i=1}^n a_{ii} x_i, \sum_{i=1}^n b_{i2} x_i, \dots, \sum_{i=1}^n b_{in} x_i \right) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$= \sum_{j=1}^n m_j x_j = \sum_{j=1}^n \sum_{i=1}^n a_{ij} x_i x_j$

$= \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i < j} a_{ij} x_i x_j$

$a_{ij} = -a_{ji}$

1. (5) $A^T A = I, B^T B = I$

$(AB)^T (AB) = B^T A^T A B = B^T I B = B^T B = I$

1. (5) $A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad A^T A = \begin{pmatrix} \alpha_1^T \alpha_1 & \dots & \alpha_1^T \alpha_n \\ \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \dots & \alpha_n^T \alpha_n \end{pmatrix} = I$

2. (1) $\mathcal{A}(x, y, z) = (x+y, x+2y, x-3z)$

$\mathcal{A}(\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$(x, y, z) = (x_1, x_2, x_3)^T$

$(x, y, z) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x}{3} \\ \frac{y}{2} \\ z \end{pmatrix}$

$\mathcal{A}(x, y, z) = \mathcal{A} \left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x}{3} \\ \frac{y}{2} \\ z \end{pmatrix} \right)$

$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x}{3} & \frac{y}{2} & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -3 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} B$

$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -3 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 0 & -3 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$

1. (5) $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -3 & -1 \end{pmatrix}, (\beta_1, \beta_2, \beta_3) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -3 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 1 & 1 & 3 & | & 0 \\ -1 & 0 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{pmatrix}$

$(\beta_1, \beta_2, \beta_3) = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}$ 线性相关, 无逆矩阵

4. (1) $(f, g) = f(w)g(u) + f(v)g(u) + f(w)g(v)$

$= g(u)f(w) + g(u)f(v) + f(w)g(v) = (f, f)$

$(f, f) = \lambda f(u)g(u) + \lambda f(v)g(v)$

$(f, f) = f(u)g(u) + f(v)g(v)$ 若 $\lambda = \frac{1}{2}$ 则 $\lambda = \frac{1}{2}$

2. $\alpha_1 = 1, \alpha_2 = \lambda$

$(\alpha_1, \alpha_2) = |1 \ 1| + |1 \ 1| = 2, \alpha_1 = \frac{1}{\sqrt{2}}$

$\beta_2 = \alpha_2 - (\alpha_1, \alpha_2)\alpha_1 = \lambda - (1 \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) \cdot \frac{1}{\sqrt{2}}$

$= \lambda - 1$

$(\beta_1, \beta_2) = (-1) \times (-1) + 0 \times 0 + (1 \times 1) = 2, \alpha_2 = \frac{1}{\sqrt{2}}$

\therefore 标准正交基为 $\left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$

5. (1) $M = \begin{pmatrix} I & A \\ A & I \end{pmatrix}, A^2 = I$

$(M^{-1})^T = A^{-1}(M^{-1})^T + A$

$P_n(\lambda) = \det(\lambda I - M) = \begin{vmatrix} (\lambda-1)I & -A \\ -A & (\lambda-1)I \end{vmatrix}$

$= \begin{vmatrix} (\lambda-1)I & -A \\ 0 & -A^2(\lambda-1)I + A \end{vmatrix} = \begin{vmatrix} (\lambda-1)I & -A \\ 0 & -I^2(\lambda-1)I + A \end{vmatrix}$

$= \det(\lambda-1)I \times \det(-(\lambda-1)I + A) \quad ((\lambda-1)I - \frac{1}{\lambda-1})I$

$= \det(-I + (\lambda-1)A)$

$= \det(-A^2 + (\lambda-1)A) = \det A \left(\frac{\lambda-1}{\lambda-1} \right)^n$

$= \det A (\lambda-1)^n$

A 不可逆, $\det A = 0, \lambda = 0$ 为特征值.

$\det(-I + (\lambda-1)A) = \det((\lambda-1)I - A) = 0$

$\lambda - 1 = \frac{1}{\lambda-1}, \lambda = 0$ 或 $\lambda = 2$

又有 $\det(\lambda-1)I = \det(-(\lambda-1)I) = \det(A - (\lambda-1)I)$

$= (\lambda-1) \det(A - (\lambda-1)I) = 0$

$\lambda = 1$ 为 -1 的特征值, $\lambda = 1$ 为特征值

\therefore 特征值 $\lambda = 0, 1, 2$

(2) $P^{-1} \begin{pmatrix} 2 & A \\ A & 2 \end{pmatrix} P = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 2 & A \\ A & 2 \end{pmatrix} P = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} P$

$\lambda \rightarrow (-\lambda - 2) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \lambda \rightarrow \begin{pmatrix} -\lambda & -A \\ -A & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$(\lambda + 2) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \lambda \rightarrow \begin{pmatrix} \lambda & A \\ A & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2, (-\lambda + A) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \lambda = 2, \begin{pmatrix} -2 & A \\ -A & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$(A - 2I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \lambda = 2, -\lambda + A \rightarrow \begin{pmatrix} A & -2I \\ -2I & A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} -Ae_1 & -Ae_2 & \dots & -Ae_n & Ae_1 & \dots & Ae_n \\ e_1 & e_2 & \dots & e_n & e_1 & \dots & e_n \end{pmatrix}$

$= \begin{pmatrix} -A & A \\ A & A \end{pmatrix}$

6. $\det A \det B = \det AB$

设 $C = AB, \forall C$ 为 n 阶方阵

设 $\det C = \left(\frac{1}{n} \text{tr } C\right)^n$

$n=1$ 时, $\det C = c_1, \text{tr } C = c_1$

$\det C = c_1 = \left(\frac{1}{1} \text{tr } C\right)^1$

$n=1$ 时, $\det C = c_1, \text{tr } C = c_1 + c_1 \dots + c_1 = nc_1$

n 阶, $C_n = \begin{pmatrix} C_{n-1} & \alpha \\ \alpha^T & C_{nn} \end{pmatrix}$

$\det C_n = \det \begin{pmatrix} C_{n-1} & \alpha \\ \alpha^T & C_{nn} \end{pmatrix} = \det \begin{pmatrix} C_{n-1} & 0 \\ \alpha^T & C_{nn} - \alpha^T C_{n-1}^{-1} \alpha \end{pmatrix}$

$= \det C_{n-1} \times \det(C_{nn} - \alpha^T C_{n-1}^{-1} \alpha)$

$C_{n-1} \geq 0, C_{nn} - \alpha^T C_{n-1}^{-1} \alpha \geq 0$

$\alpha^T C_{n-1}^{-1} \alpha \geq 0, C_{nn} \geq 0$ 都有 $\det C_{nn} - \alpha^T C_{n-1}^{-1} \alpha \geq 0$

$\therefore \det C_n = \det C_{n-1} \times (C_{nn} - \alpha^T C_{n-1}^{-1} \alpha)$

$\leq C_{n-1} \times C_{nn} = C_n$

$\therefore \det C_n \leq C_n \leq \left(\frac{\text{tr } C_n}{n}\right)^n = \left(\frac{1}{n} \text{tr } C\right)^n$