08～09期末

2021年12月29日 星期三 下午10:04

中国科学技术大学 2008—2009 学年第一学期 线性代数期末考试

1. (35 分) 填空题.

(1) 设 A, B 均为 n 阶方阵, $|A| = -3$, $|B| = -2$, 则 $A^* =$ ____, $|B^{-1}| =$ ____.

(2) $\alpha = (1,2,3)^T$, $\beta = (1,-1,1)$, 则 $(\alpha\beta)^2 =$ ____.

(3) 设 $A = \mathrm{diag}\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{7}\right)$, $A^{-1}BA = BA + 6AA$, 则 $|B^*| =$ ____.

(4) 设 $\alpha_1 = (1,0,0)^T$, $\alpha_2 = (a,1,0)^T$, $\alpha_3 = (c,b,1)^T$, 则三阶方阵 $(\alpha_1, \alpha_2, \alpha_3)$ 的逆矩阵为 ____.

(5) 设 $n \geq 2$, $\beta_1 = (1,2,\cdots,n)^T$, $\beta_2 = (2,3,\cdots,n+1)^T$, $\cdots$, $\beta_n = (n, n+1, \cdots, 2n-1)^T$, 则 $\mathrm{rank}(\beta_1, \cdots, \beta_n) =$ ____.

(6) 设 $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, 则 $P_1^{99} A P_2^{99} =$ ____.

(7) 设 A 为三阶矩阵, 特征值分别为 3, 2, 1, 其对应特征向量分别为 $\alpha_1, \alpha_2, \alpha_3$. 记 $P = (\alpha_1, \alpha_2, \alpha_3)$, 则 $P^{-1}AP =$ ____.

(8) n 维实向量 X, Y 满足 $|X+Y|^2 = |X|^2 + |Y|^2$, 则 X 与 Y ____.

(9) 若上三角阵 A 是正交矩阵, 则 A 主对角线上的元素 $a_{ii} =$ ____.

(10) 设三阶矩阵 A 的特征值为 2, 4, 5, $B = A^{-1} - 2I$, 则 B 的特征值为 ____.

(11) $\lambda =$ ____ 时, 方程 $x^2 + (\lambda+1)y^2 + \lambda^2 z^2 + 2x + 2z = 1$ 表示椭球面.

2. (15 分) 设 $V = M_2(\mathbb{R}) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$, V 上的线性变换 $\mathscr{A}\alpha = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$.

(1) 验证 (E): $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ 是 V 的一组基.

(2) 求 $\alpha = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ 在 (E) 下的坐标 X.

(3) 求线性变换 $\mathscr{A}$ 在 (E) 下的表示矩阵.

3. (8 分) 设 $\alpha_1, \cdots, \alpha_{n-1}$ 是 $\mathbb{R}^n$ 中的线性无关组, β_1, β_2 都与 $\alpha_1, \cdots, \alpha_{n-1}$ 正交. 证明: β_1 和 β_2 线性相关.

4. (18 分) 已知 $f(x,y,z) = 4x^2 - 6y^2 - 6z^2 - 4yz$.

(1) 写出 $f(x,y,z)$ 的二次型矩阵.

(2) 用正交变换化二次型 $f(x,y,z)$ 为标准型.

(3) 判断曲面 $f(x,y,z) = 1$ 的类型.

5. (16 分) 已知非齐次方程组 $\begin{cases} x_1 + x_2 + x_3 + x_4 = -1 \\ 4x_1 + 3x_2 + 5x_3 - x_4 = -1 \\ ax_1 + x_2 + 3x_3 + bx_4 = 1 \end{cases}$ 有 3 个线性无关的解.

(1) 证明: 方程组系数矩阵 A 的秩为 2.

(2) 求 a, b 的值及方程组的通解.

6. (8 分) 设 A 为 n 阶实矩阵, $A^2 = kA$. 证明: A 可相似对角化.

求逆公式

1.(1) $AA^* = |A|I$　　$|A| = -3$.

$BB^* = |B|I$,　$B^{-1} = \frac{B^*}{|B|}$.　$|B^{-1}| = \frac{|B^*|}{|B|^n}$.

(2) $\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\beta = (1,-1,1)$.

$\alpha\beta = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(1,-1,1) = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ 4 & -4 & 4 \\ 6 & -6 & 6 \end{pmatrix}$

(3) $A = \begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{4} & \\ & & \frac{1}{7} \end{pmatrix}$　$A^{-1}BA = BA + 6AA$

$A^{-1}BAA^{-1} = BAA^{-1} + 6AAA^{-1}$

$AB = B + 6A$.

$AB - B = 6A$.　$(A-I)B = 6A$,　$B = 6(A-I)^{-1}A$.

$A - I = \begin{pmatrix} -\frac{2}{3} & & \\ & -\frac{3}{4} & \\ & & -\frac{6}{7} \end{pmatrix}$. $(A-I)^{-1} = \begin{pmatrix} -\frac{3}{2} & & \\ & -\frac{4}{3} & \\ & & -\frac{7}{6} \end{pmatrix}$

$B = \begin{pmatrix} -9 & & \\ & -8 & \\ & & -7 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{4} & \\ & & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} -3 & & \\ & -2 & \\ & & -1 \end{pmatrix}$

$BB^* = |B|I$.　$B^* = |B|B^{-1}$.　$|B^*| = \det(|B|B^{-1})$

$|B| = -6$.　$B^{-1} = \begin{pmatrix} -\frac{1}{3} & & \\ & -\frac{1}{2} & \\ & & -1 \end{pmatrix}$　$= |B|^3 \times |B^{-1}|$

$= -216 \times -\frac{1}{6} = 36$.

(4) $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$　$\det(A) = 1$.　$\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{pmatrix}$

(5) $\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & n+1 \\ 3 & 4 & 5 & \cdots & n+1 & n+2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n & n+1 & \cdots & 2n-3 & 2n-2 \\ n & n+1 & n+2 & \cdots & 2n-2 & 2n-1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$　rank $A = 1$.

(6) 看. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{11}+a_{31} & 2a_{12}+a_{32} & 2a_{13}+a_{33} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$2r_1 + r_3$.　r_1 & r_3.

$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \to \begin{pmatrix} 2\alpha_1+\alpha_3 \\ \alpha_2 \\ \alpha_1 \end{pmatrix} \to \begin{pmatrix} 2(2\alpha_1+\alpha_3)+\alpha_1 \\ \alpha_2 \\ 2\alpha_1+\alpha_3 \end{pmatrix} = \begin{pmatrix} 5\alpha_1+2\alpha_3 \\ \alpha_2 \\ 2\alpha_1+\alpha_3 \end{pmatrix}$

$\xrightarrow{99} \begin{pmatrix} (2^{99}+1)\alpha_1 + 2^{98}\alpha_3 \\ \alpha_2 \\ 2^{98}\alpha_1 + 2^{97}\alpha_3 \end{pmatrix}$.

特征值性质

(7) $Ax = \lambda x$.　$A(\alpha_1^T, \alpha_2^T, \alpha_3^T) = \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1 \end{pmatrix}(\alpha_1^T, \alpha_2^T, \alpha_3^T)$

(8) $|X+Y|^2 = |X|^2 + |Y|^2$.　$XY = 0$, 正交.

(9) ± 1.

特征值

(10) $p_A(\lambda) = (\lambda-2)(\lambda-4)(\lambda-5)$　$(\lambda I - A)$

$\det(\lambda I - A) = p_A(\lambda) = (\lambda-2)(\lambda-4)(\lambda-5)$.

$B = A^{-1} - 2I$.　$B + 2I = A^{-1}$.　$(B+2I)^{-1} = A$

$= \det(\lambda I - (B+2I)^{-1})$

$\det(\lambda I - B) = p_B(\lambda) = \det(\lambda I - A^{-1} + 2I)$

$= \det((\lambda+2)I - A^{-1})$

二次曲面

(11) $x^2 + (\lambda+1)y^2 + \lambda^2 z^2 + 2x + 2z - 1 = 0$

$(x\ y\ z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$　$\lambda = 1$?

线性变换

2.(1) $V = M_2(\mathbb{R}) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$.　$\mathscr{A}\alpha = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$

对 $\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4$

$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_2 \\ \lambda_2 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_3 & 0 \\ 0 & -\lambda_3 \end{pmatrix} + \begin{pmatrix} 0 & -\lambda_4 \\ \lambda_4 & 0 \end{pmatrix}$

$= \begin{pmatrix} \lambda_1+\lambda_3 & \lambda_2-\lambda_4 \\ \lambda_2+\lambda_4 & \lambda_1-\lambda_3 \end{pmatrix} = 0$ 方程,

有 $\begin{cases} \lambda_1+\lambda_3 = 0 \\ \lambda_1-\lambda_3 = 0 \\ \lambda_2+\lambda_4 = 0 \\ \lambda_2-\lambda_4 = 0 \end{cases}$, rank $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = 4$.

只有零解, 线性无关.

又 $\dim V = 4$, ∴ 是一组基.

(2)

3. $\alpha_1 \cdots \alpha_{n-1}$ 线性无关,

可扩充为 $\mathbb{R}^n$ 的基 $\alpha_1 \cdots \alpha_{n-1}\ \alpha_n$.

对 $\forall \alpha_i$, $1 \leq i \leq n-1$, 有 $(\beta_1, \alpha_i) = 0$, $(\beta_2, \alpha_i) = 0$

对 $\forall \alpha \in V$, 有 $\alpha = \sum_{i=1}^{n} \lambda_i \alpha_i$.

$\sum_{i=1}^{n} \lambda_i(\beta_1, \alpha_i) = (\beta_1, \sum_{i=1}^{n} \lambda_i \alpha_i) = (\beta_1, \alpha)$

$= \sum_{i=1}^{n-1}(\beta_1, \alpha_i) + (\beta_1, \alpha_n) = (\beta_1, \alpha_n)$.

若 $(\beta_1, \alpha) = 0$, 则 β_1 与 α 正交, β_1 与 α_n 正交.

则 $\beta_1 \cdot \alpha_1 \cdots \alpha_n$ 均线性无关, 构成 $\mathbb{R}^n$ 中的

极大无关组, 则 $\alpha_1 \cdots \alpha_n$ 不是基, 矛盾.

因此 $(\beta_1, \alpha) \neq 0$, $(\beta_1, \alpha_n) \neq 0$.

∴ β_1 与 $\alpha_1 \cdots \alpha_n$ 线性相关,

$\beta_1 = \lambda_1\alpha_1 \cdots \lambda_n\alpha_n$, $\exists \lambda_i \neq 0$.

又 ∵ β_1 与 $\alpha_1 \cdots \alpha_{n-1}$ 正交, ∴ $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$.

∴ $\lambda_n \neq 0$, 即 $\beta_1 = \lambda_n \alpha_n$. β_1 与 α_n 线性相关.

同理, β_2 与 α_n 线性相关, $\beta_2 = \lambda_n' \alpha_n$.

∴ $\beta_1 = \frac{\lambda_n}{\lambda_n'}\beta_2$, 线性相关.

4.(1) $f(x,y,z) = 4x^2 - 6y^2 - 6z^2 - 4yz$

$= (x\ y\ z)\begin{pmatrix} 4 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & -6 \end{pmatrix}$

(2) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-\frac{1}{3}c_3 + c_2]{-\frac{1}{3}r_3 + r_2} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -6+\frac{2}{3} & -2 \\ 0 & 0 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -\frac{16}{3} & 0 \\ 0 & 0 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}$

(3) $4x^2 - 6y^2 - 6z^2 - 4yz = 1$.

取 $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}$, $P^TAP = \begin{pmatrix} 4 & & \\ & -\frac{16}{3} & \\ & & -6 \end{pmatrix}$,

得 $4\tilde{x}^2 - \frac{16}{3}\tilde{y}^2 - 6\tilde{z}^2 = 1$. 单叶双曲?

5.(1) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 5 & -1 \\ a & 1 & 3 & b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -5 \\ 0 & -a+1 & -a+3 & -a+b \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -5 \\ 0 & 0 & -2a+4 & 4a+b-5 \end{pmatrix}$

有三组线性无关解 ⇒ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = X_1 t_1 + X_2 t_2 + X_3$

即有两个自由变量, 解空间维数为 2.

∴ rank $A = 2$

(2) ∴ $-2a + 4 = 0$, $4a + b - 5 = 0$,　$a = 2$, $b = -3$.

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -5 \end{pmatrix}$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \\ 1 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} \\ -5 \\ 0 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} \\ \\ \\ \end{pmatrix}$.

$x_3 = t_1$, $x_4 = t_2$

6. A 可相似对角化

⇔ ∃可逆 P, 使 $P^{-1}AP = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$.

$A^2 = kA$. 取 $A = (\alpha_1\ \alpha_2 \cdots \alpha_n)$, α_i 为列向量.

$A(\alpha_1\ \alpha_2 \cdots \alpha_n) = k(\alpha_1 \cdots \alpha_n)$

$n = 1$ 时, 显然成立.

设阶数 $n-1$ 时, ∃ P_{n-1}, 使 $P_{n-1}^{-1}A_{n-1}P_{n-1} = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_{n-1} \end{pmatrix}$.

$A_{n-1} = P_{n-1}\begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_{n-1} \end{pmatrix}P_{n-1}^{-1}$.

则当阶数 n 时, $A_n = \begin{pmatrix} A_{n-1} & C_2 \\ C_1 & a_{nn} \end{pmatrix}$.

$A_nA_n = \begin{pmatrix} A_{n-1} & C_2 \\ C_1 & a_{nn} \end{pmatrix}\begin{pmatrix} A_{n-1} & C_2 \\ C_1 & a_{nn} \end{pmatrix}$

$= \begin{pmatrix} A_{n-1}A_{n-1} + C_2C_1 & A_{n-1}C_2 + a_{nn}C_2 \\ C_1A_{n-1} + a_{nn}C_1 & a_{nn}^2 + C_1C_2 \end{pmatrix}$

$A_{n-1}A_{n-1} = k_{n-1}A_{n-1}$.

$= \begin{pmatrix} k_{n-1}A_{n-1} + C_2C_1 & (A_{n-1} + a_{nn}I)C_2 \\ (A_{n-1} + a_{nn}I)C_1 & a_{nn}^2 + C_1C_2 \end{pmatrix}$

$= k_n \begin{pmatrix} A_{n-1} & C_2 \\ C_1 & a_{nn} \end{pmatrix}$

∴ $A_{n-1} + a_{nn}I = k_n$.