课后题&补充题

2021年12月28日 星期二　　上午10:26

1. (1) $Q(x_1,x_2,x_3) = -x_1^2+x_2^2+2x_3^2+4x_1x_2+3x_1x_3+4x_2x_3$

$= (x_1\ x_2\ x_3)\ A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$A=\begin{pmatrix}-1&2&\frac{3}{2}\\2&1&2\\\frac{3}{2}&2&2\end{pmatrix}$

(2) $Q(x_1,x_2,x_3) = 2x_1^2+0x_2^2+0x_3^2+2x_1x_2+0x_1x_3-x_2x_3$

$= (x_1\ x_2\ x_3)\ A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$A=\begin{pmatrix}2&1&0\\1&0&-\frac{1}{2}\\0&-\frac{1}{2}&0\end{pmatrix}$

(3) $Q(x_1,x_2,x_3,x_4) = x_1x_2+3x_1x_3+5x_1x_4+6x_2x_3+7x_2x_4+10x_3x_4$

$A=\begin{pmatrix}0&\frac{1}{2}&\frac{3}{2}&\frac{5}{2}\\\frac{1}{2}&0&3&\frac{7}{2}\\\frac{3}{2}&3&0&5\\\frac{5}{2}&\frac{7}{2}&5&0\end{pmatrix}$

(4) $Q(x_1\cdots x_n) = (x_1-x_3)^2+(x_2-x_4)^2+\cdots+(x_{n-2}-x_n)^2$

$= x_1^2+x_2^2+\cdots+x_n^2$

$-2x_1x_3-2x_2x_4-\cdots-2x_{n-2}x_n$

$A=\begin{pmatrix}1&&-1&&\\0&1&0&&\\-1&0&1&&\\&&&\ddots&\\&&&&1\end{pmatrix}$　$-\frac{1}{2}-\frac{(-1)^{m+n+1}}{2}$

2. (1) $Q(x_1,x_2,x_3) = x_1^2+2x_2^2+4x_3^2$

$+6x_1x_2+10x_1x_3+12x_2x_3$.

(2) $Q(x_1,x_2,x_3) = ax_1^2+ax_2^2+ax_3^2$

$+2bx_1x_2+2bx_2x_3$.

(3) $Q(x_1,x_2,x_3,x_4) = -2x_1x_2+4x_1x_3+6x_2x_3$

$-6x_1x_4-4x_2x_4+4x_3x_4$.

3. (1) $\begin{pmatrix}1&-2&2&1&0&0\\-2&4&-4&0&1&0\\2&-4&4&0&0&1\end{pmatrix}$

$\to\begin{pmatrix}1&0&0&1&0&0\\-2&0&0&0&1&0\\2&0&0&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&1&0&0\\0&0&0&2&1&0\\0&0&0&-2&0&1\end{pmatrix}$

$P^T=\begin{pmatrix}1&2&-2\\0&1&0\\0&0&1\end{pmatrix}$

$Q(x_1,x_2,x_3) = (x_1\ x_2\ x_3)\ A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$= (y_1\ y_2\ y_3)\underline{P\,A\,P^T}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$= (y_1\ y_2\ y_3)\ \underline{B}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$A=\begin{pmatrix}1&-2&2\\-2&4&-4\\2&-4&4\end{pmatrix}$　$\begin{pmatrix}1&0&0\\2&1&0\\-2&0&1\end{pmatrix}\begin{pmatrix}1&-2&2\\-2&4&-4\\2&-4&4\end{pmatrix}\begin{pmatrix}1&2&-2\\0&1&0\\0&0&1\end{pmatrix}$

$=\begin{pmatrix}1&-2&2\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&2&-2\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$Q(y_1,y_2,y_3) = y_1^2$.

(3) $\begin{cases}x_1=y_1-y_2\\x_2=y_1+y_2\\x_3=y_3\end{cases}\Rightarrow Q(y_1,y_2,y_3)$

$=6(y_1^2-y_2^2)+6(y_1-y_2)y_3+6(y_1+y_2)y_3$

$=6y_1^2-6y_2^2+12y_1y_3$

$=6y_1^2+12y_1y_3+6y_3^2-6y_2^2-6y_3^2$

$=6(z_1)^2-6z_2^2-6z_3^2$

$=\begin{pmatrix}6&0&6\\0&-6&0\\6&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}6&0&6\\0&-6&0\\0&0&-6\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}6&0&0\\0&-6&0\\0&0&-6\\1&0&-1\\0&1&0\\0&0&1\end{pmatrix}$

$Q(y_1,y_2,y_3) = 6y_1^2-6y_2^2-6y_3^2$.

4. (1) $Q = x_1x_2+x_2x_3+x_3x_4+x_4x_1$

$=(y_1+y_2)(y_1-y_2)+(y_1-y_2)y_3+y_3y_4+y_4(y_1+y_2)$

$=y_1^2-y_2^2+\underline{y_1y_3}-y_2y_3+y_3y_4+y_1y_4+y_2y_4$　$=(y_1+\frac{1}{2}y_3+\frac{1}{2}y_4)^2-(y_2+\frac{1}{2}y_3+\frac{1}{2}y_4)^2=z_1^2-z_2^2$

$=z_1^2-z_2^2+z_1(z_3+z_4)+z_2(z_3+z_4)+z_3^2-z_4^2$

$+z_1(z_3-z_4)+z_2(z_3-z_4)$

$=z_1^2-z_2^2+z_3^2-z_4^2+2z_1z_3+2z_2z_3$

$=(z_1^2+2z_1z_3+z_3^2)-z_2^2+2z_2z_3^2-z_3^2-z_4^2+z_3^2$

$=(z_1+z_3)^2-(z_2-z_3)^2-z_4^2+z_3^2$

$=a_1^2-a_2^2+a_3^2-a_4^2$

$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\begin{pmatrix}1&1&&\\1&-1&&\\&&1&1\\&&1&-1\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix}$　$\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}1&0&1&1\\1&0&1&-1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$

$a_1=z_1+z_3$.　$a_2=z_2-z_3$.　$a_3=z_3$.　$a_4=z_4$

$=\begin{pmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&-1&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&0&0\\1&-1&0&0\\1&-1&1&0\\1&-1&1&-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\\a_4\end{pmatrix}$

$x=Pa$.　$P=\begin{pmatrix}1&1&-1&-1\\1&-1&-1&1\\0&0&1&1\\0&0&1&-1\end{pmatrix}$

$Q=(x^T)Ax=(Pa)^TA(Pa)$

$B=P^TAP=\begin{pmatrix}1&1&1&1\\1&-1&-1&-1\\0&0&1&1\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1&1&0&0\\1&-1&0&0\\1&-1&1&0\\1&-1&1&-1\end{pmatrix}$

$=\begin{pmatrix}1&2&2&1\\-1&0&-2&-1\\0&1&1&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}1&1&0&0\\1&-1&0&0\\1&-1&1&0\\1&-1&1&-1\end{pmatrix}=\begin{pmatrix}6&-4&3&-1\\-4&2&-3&1\\3&-3&2&-1\\-1&1&-1&0\end{pmatrix}$

17. $A=(a_{ij})$. n阶正定. $\det(A)\leq a_{11}a_{22}\cdots a_{nn}$

$n=1$, $A=(a_{11})>0$, $\det A=a_{11}$ ✓

$n=k$, $A=\begin{pmatrix}a_{11}&\cdots&a_{1k}\\\vdots&\ddots&\vdots\\a_{k1}&\cdots&a_{kk}\end{pmatrix}$, 假设 $\det A\leq a_{11}\cdots a_{kk}$ 成立.

$n=k+1$ 时, $A=\begin{pmatrix}a_{11}&\cdots&a_{1k}&a_{1,k+1}\\\vdots&\ddots&\vdots&\vdots\\a_{k1}&\cdots&a_{kk}&a_{k,k+1}\\a_{k+1,1}&\cdots&a_{k+1,k}&a_{k+1,k+1}\end{pmatrix}$　T27. 不等式.

$A=P^TP$
上三角矩阵方法

$\det A=a_{k+1,k+1}\det(A_k)\leq a_{11}\cdots a_{kk}a_{k+1,k+1}$

P. (1) $Q=x_1^2+2x_2^2+4x_3^2+2x_1x_2-4x_2x_3$

$=x_1^2+2x_1x_2+x_2^2+x_2^2-4x_2x_3+4x_3^2$

$=(x_1+x_2)^2+(x_2-2x_3)^2=y_1^2+y_2^2$

$y_1=x_1+x_2$, $y_2=x_2-2x_3$　正定.

(2) $Q=-5x_1^2-6x_2^2-4x_3^2+4x_1x_2-4x_1x_3$

12. (1) $Q=2x_1^2+x_2^2+x_3^2+2x_1x_2+tx_1x_3$

$=(x_1\ x_2\ x_3)\begin{pmatrix}2&1&\frac{t}{2}\\1&1&0\\\frac{t}{2}&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$=x_1^2+2x_1x_2+x_2^2+x_1^2+tx_1x_3+x_3^2$

$=(x_1+x_2)^2+(x_1+\frac{t}{2}x_3)^2+(1-\frac{t^2}{4})x_3^2$

$\begin{cases}y_1=x_1+x_2\\y_2=x_1+\frac{t}{2}x_3\\y_3=x_3\end{cases}$　$=y_1^2+y_2^2+(1-\frac{t^2}{4})y_3^2$

$1-\frac{t^2}{4}>0$, $\frac{t^2}{4}<1$, $t^2<2$.

(2) $Q=x_1^2+2x_2^2+3x_3^2+tx_1x_2+tx_1x_3+x_2x_3$

$=(x_1\ x_2\ x_3)\begin{pmatrix}1&\frac{t}{2}&\frac{t}{2}\\\frac{t}{2}&2&\frac{1}{2}\\\frac{t}{2}&\frac{1}{2}&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$=x_1^2+tx_1x_2+\frac{t^2}{4}x_2^2+(2-\frac{t^2}{4})x_2^2+3x_3^2+tx_1x_3+x_2x_3$

$\begin{pmatrix}1&\frac{t}{2}&\frac{t}{2}\\\frac{t}{2}&2&\frac{1}{2}\\\frac{t}{2}&\frac{1}{2}&3\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&\frac{t}{2}\\\frac{t}{2}&2-\frac{t^2}{4}&\frac{t}{2}\\\frac{t}{2}&\frac{1}{2}-\frac{t^2}{4}&3\\1&-\frac{t}{2}&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&\frac{t}{2}\\0&2-\frac{t^2}{4}&\frac{1}{2}-\frac{t^2}{4}\\\frac{t}{2}&\frac{1}{2}-\frac{t^2}{4}&3\\1&-\frac{t}{2}&0\\0&1&0\\0&0&1\end{pmatrix}$

$\to\begin{pmatrix}1&0&0\\0&2-\frac{t^2}{4}&\frac{1}{2}-\frac{t^2}{4}\\0&\frac{1}{2}-\frac{t^2}{4}&3-\frac{t^2}{4}\\1&-\frac{t}{2}&-\frac{t}{2}\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&2-\frac{t^2}{4}&0\\0&0&\frac{(t^2-2)^2+(12-t^2)(t^2-8)}{4(t^2-8)}\\1&-\frac{t}{2}&-\frac{t}{2}\times\left(1-\frac{\frac{1}{2}-\frac{t^2}{4}}{2-\frac{t^2}{4}}\right)\\0&1&-\frac{\frac{1}{2}-\frac{t^2}{4}}{2-\frac{t^2}{4}}\\0&0&1\end{pmatrix}$

$\left(2-\frac{t^2}{4}\right)\times\left(\frac{\frac{1}{2}-\frac{t^2}{4}}{\left(2-\frac{t^2}{4}\right)}\right)=-\left(\frac{1}{2}-\frac{t^2}{4}\right)$.

$\left(\frac{1}{2}-\frac{t^2}{4}\right)\left(-\frac{\frac{1}{2}-\frac{t^2}{4}}{2-\frac{t^2}{4}}\right)=\left(\frac{1}{2}-\frac{t^2}{4}\right)\left(-\frac{2-t^2}{8-t^2}\right)$

$=\left(\frac{t^2-2}{4}\right)\times\left(\frac{t^2-2}{t^2-8}\right)=\frac{(t^2-2)^2}{4}\times\frac{1}{(t^2-8)}$

$\frac{(t^2-2)^2}{4(t^2-8)}+\frac{12-t^2}{4}$

$\therefore 2-\frac{t^2}{4}>0$.

$\frac{(t^2-2)^2+(12-t^2)(t^2-8)}{4(t^2-8)}>0$

22. (1) $4x^2-6y^2-6z^2-4yz-4x+4y+4z-5=0$

$\begin{pmatrix}4&0&0\\0&-6&-2\\0&-2&-6\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}4&0&0\\0&-6&0\\0&0&-\frac{16}{3}\\1&0&0\\0&1&-\frac{1}{3}\\0&0&1\end{pmatrix}$　$-2\times(-\frac{1}{3})=\frac{2}{3}$　$-6=\frac{2}{3}-\frac{18}{3}$

---

1. (2) $Q(x_1,x_2,x_3)=-4x_1x_2+2x_1x_3+2x_2x_3$

$=-4(y_1+y_2)(y_1-y_2)+2(y_1+y_2)y_3+2(y_1-y_2)y_3$

$=-4(y_1^2-y_2^2)+4y_2y_3=-4y_1^2+4y_2^2+4y_2y_3$.

$=-4y_1^2+4y_2^2+4y_2y_3+y_3^2-y_3^2$

$=-4y_1^2+(2y_2+y_3)^2-y_3^2$

$=-4z_1^2+z_2^2-z_3^2$.　$\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1&1&0\\1&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}1&1&0\\1&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&1&-2\\0&-1&2\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$

$=\begin{pmatrix}2&1&-2\\2&-1&2\\0&-1&2\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$.

$Q=(x_1\ x_2\ x_3)\begin{pmatrix}0&-2&1\\-2&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$　$=xAx$

$=(Pz)^TAPz$

$=z^TP^TAPz$.

$=(z_1\ z_2\ z_3)\begin{pmatrix}2&2&0\\1&-1&-1\\-2&2&2\end{pmatrix}\begin{pmatrix}0&-2&1\\-2&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}2&1&-2\\2&-1&2\\0&-1&2\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$

2. $Q=ax_1^2+bx_2^2+ax_3^2+2cx_1x_3$

$=(x_1\ x_2\ x_3)\begin{pmatrix}a&0&c\\0&b&0\\c&0&a\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$\begin{pmatrix}a&0&c\\0&b&0\\c&0&a\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}a&0&0\\0&b&0\\0&0&a-\frac{c^2}{a}\\1&0&-\frac{c}{a}\\0&1&0\\0&0&1\end{pmatrix}$　$a>0$
$b>0$
$a>\frac{c^2}{a}$, $c^2<a^2$

顺序主子式. $a>b$. $ab>0$. $(a^2-c^2)b>0$

3. $A_1<0$.

4. $Q=\lambda x^2+(\lambda+3)y^2+\lambda z^2-6xy-2xz+4yz$

$=(x\ y\ z)\begin{pmatrix}\lambda&-2&-1\\-2&\lambda+3&2\\-1&2&\lambda\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\lambda\neq0$ 时

$\begin{pmatrix}\lambda&-2&-1\\-2&\lambda+3&2\\-1&2&\lambda\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}\lambda&0&0\\-2&-\frac{4}{\lambda}+\lambda+3&-\frac{2}{\lambda}+2\\-1&-\frac{2}{\lambda}+2&-\frac{1}{\lambda}+\lambda\\1&\frac{2}{\lambda}&\frac{1}{\lambda}\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\lambda&0&0\\0&-\frac{4}{\lambda}+\lambda+3&-\frac{2}{\lambda}+2\\0&-\frac{2}{\lambda}+2&-\frac{1}{\lambda}+\lambda\\1&\frac{2}{\lambda}&\frac{1}{\lambda}\\0&1&0\\0&0&1\end{pmatrix}$

$\to\begin{pmatrix}\lambda&0&0\\0&&\\&&\end{pmatrix}$

$\left(-\frac{1}{\lambda}+\lambda\right)\times\frac{-\frac{2}{\lambda}+2}{-\frac{1}{\lambda}+\lambda}$

$\left(-\frac{2}{\lambda}+2\right)\left(-\frac{-\frac{2}{\lambda}+2}{\frac{1}{\lambda}+\lambda}\right)+-\frac{4}{\lambda}+\lambda+3$

$=\left(-\frac{2}{\lambda}+2\right)\left(-\frac{-2+2\lambda^2}{1+\lambda^2}\right)$

半负定 $\begin{cases}|S|=0\\|S|\neq0\Rightarrow S负定, -S正定.\end{cases}$

5. A n阶实对称正定, 证明 A^{-1} 正定.

A 相合于 I_n. $\exists P$可逆 使 $P^TAP=I_n$

$P^TA=P^{-1}$.　$P^T=P^{-1}A^{-1}$.　$I_n=P^{-1}A^{-1}(P^T)^{-1}$.

$A^{-1}=(A^{-1})^TA(A^{-1})$. 与正定矩阵合同

25.

6. S 实对称正定. $\exists$上三角阵T 使 $S=T^TT$.

$n=1$, 显然成立.

$n=k-1$, 假设成立.

$Q=(x_1\cdots x_{k-1})S_{k-1}\begin{pmatrix}x_1\\\vdots\\x_{k-1}\end{pmatrix}=X^TSX$

$\overset{X=PY}{=}y_1^2+\cdots+y_{k-1}^2$

$=Y^TP^T\begin{pmatrix}I_r&&\\&-I_r&\\&&0\end{pmatrix}PY$.

有 $S_{k-1}=P^T\begin{pmatrix}I_r&&\\&-I_s&\\&&0\end{pmatrix}P=T^T\begin{pmatrix}I_r&&\\&I_s&\\&&0\end{pmatrix}T$

$\therefore$ 必有 $s=0$

$n=k$, $Q=(x_1\cdots x_k)S_k\begin{pmatrix}x_1\\\vdots\\x_k\end{pmatrix}$

S 相合于 I_n.

$\exists$可逆P, 使 $S=P^TP$.

对可逆P, $\exists$正交阵T和上三角阵Q,

使 $P=TQ$. (QR分解, 正交化法).

$S=Q^TT^TTQ$

7. A, B n阶实对称, A正定.

$\exists$可逆P, 使 $P^TAP=I_n$, P^TBP为对角阵.

取可逆P_1, $P_1^TBP_1=D$.

$P^TAP=I_n$. $P=P_1T$.

$T^TP_1^TAP_1T=T^TT=I_n$. (T正交).

8. $A>0$. 则$\exists$正定阵C, 使 $A=C^2$.

$\exists$正交阵T使 $A=T^T\begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_n\end{pmatrix}T$.

$A=T^T\begin{pmatrix}\sqrt{\lambda_1}&&\\&\ddots&\\&&\sqrt{\lambda_n}\end{pmatrix}^2T$.

取 $C=T^T\begin{pmatrix}\sqrt{\lambda_1}&&\\&\ddots&\\&&\sqrt{\lambda_n}\end{pmatrix}T$ 即可.

$Q(x_1,x_2,x_3)=3x_1^2+8x_2^2-x_3^2-12x_2x_3$

$\underline{y=Px}\Rightarrow x=P^Ty$.

$\lambda=11$.　$\vec{x}_1=\begin{pmatrix}0\\-2\\1\end{pmatrix}/\sqrt{5}$.

$\lambda=3$.　$\vec{x}_2=\begin{pmatrix}1\\0\\0\end{pmatrix}$　$\Rightarrow$不同λ.
不需要正交化

$\lambda=-4$.　$\vec{x}_3=\begin{pmatrix}0\\1\\2\end{pmatrix}/\sqrt{5}$

---

1. 求标准形.

(1) $\begin{pmatrix}2&4&-2\\4&5&-1\\-2&-1&0\end{pmatrix}$

$3\times\frac{3}{2}=\frac{9}{2}$　$-3=\frac{3}{2}-\frac{6}{2}=\frac{3}{2}$

$\begin{pmatrix}2&4&-2\\4&5&-1\\-2&-1&0\\1&&\\&1&\\&&1\end{pmatrix}\to\begin{pmatrix}2&0&-2\\0&-3&3\\-2&3&0\\1&-2&\\&1&\\&0&1\end{pmatrix}\to\begin{pmatrix}2&0&0\\0&-3&3\\0&3&-2\\1&-2&1\\&1&\\&&1\end{pmatrix}\to\begin{pmatrix}2&0&0\\0&\frac{3}{2}&0\\0&0&-2\\1&-\frac{1}{2}&1\\&1&\\&\frac{3}{2}&1\end{pmatrix}$

$P^TAP=\begin{pmatrix}1&&\\-\frac{1}{2}&1&\frac{3}{2}\\1&&1\end{pmatrix}\begin{pmatrix}2&4&-2\\4&5&-1\\-2&-1&0\end{pmatrix}\begin{pmatrix}1&-\frac{1}{2}&1\\&1&\\&\frac{3}{2}&1\end{pmatrix}=\begin{pmatrix}2&&\\&\frac{3}{2}&\\&&-2\end{pmatrix}$

(2) $\begin{pmatrix}1&2&3\\2&3&4\\3&4&6\end{pmatrix}$

$\begin{pmatrix}1&2&3\\2&3&4\\3&4&6\\1&&\\&1&\\&&1\end{pmatrix}\to\begin{pmatrix}1&0&3\\0&-1&-2\\3&-2&6\\1&-2&\\&1&\\&&1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&-1&-2\\0&-2&-3\\1&-2&-3\\&1&\\&&1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\\1&-2&1\\&1&-2\\&&1\end{pmatrix}$

$\to\begin{pmatrix}2&0&0\\0&-3&0\\0&0&1\\1&-2&-1\\&1&1\\&&1\end{pmatrix}$

定理证明.

各阶顺序主子式均大于零. 正定.

$n=1$, $a_{11}>0$. 正定.

$n-1$ 时. $A_{n-1}=\begin{pmatrix}a_{11}&\cdots&a_{1,n-1}\\\vdots&&\vdots\\a_{n-1,1}&\cdots&a_{n-1,n-1}\end{pmatrix}$ 各阶>0, 正定.

$\exists P_{n-1}$, $P_{n-1}^TA_{n-1}P_{n-1}=I_{n-1}$

$A=\begin{pmatrix}A_{n-1}&C\\C^T&a_{nn}\end{pmatrix}$

$\begin{pmatrix}P_{n-1}^T&0\\-C^TA_{n-1}^{-1}&1\end{pmatrix}\begin{pmatrix}A_{n-1}&C\\C^T&a_{nn}\end{pmatrix}\begin{pmatrix}P_{n-1}&-A_{n-1}^{-1}C\\0&1\end{pmatrix}=\begin{pmatrix}I_{n-1}&0\\0&-C^TA_{n-1}^{-1}C+a_{nn}\end{pmatrix}$　$-A_{n-1}C$　$P_{n-1}^TA_{n-1}P_{n-1}$

$\begin{pmatrix}P_{n-1}^TA_{n-1}&P_{n-1}^TC\\0&-C^TA_{n-1}^{-1}C+a_{nn}\end{pmatrix}\begin{pmatrix}&\\&\end{pmatrix}=\begin{pmatrix}I_{n-1}&0\\0&a\end{pmatrix}$

$P=\begin{pmatrix}P_{n-1}&-A_{n-1}C\\0&1\end{pmatrix}\begin{pmatrix}I_{n-1}&\\&\frac{1}{\sqrt{a}}\end{pmatrix}$

T25

$n=1$ 时, 显然成立.

$n-1$ 时假设成立. (S_{n-1}正定. $S_{n-1}=T_{n-1}^TT_{n-1}$)

对 $S_n=\begin{pmatrix}S_{n-1}&\alpha^T\\\alpha&s_{nn}\end{pmatrix}$ 正定.

则其中 S_{n-1} 正定.

$\therefore\exists$上三角阵 $T_n=\begin{pmatrix}I_{n-1}&-S_{n-1}^{-1}\alpha^T\\0&1\end{pmatrix}$,

使 $T_n^TS_nT_n=\begin{pmatrix}S_{n-1}&\\&d_n\end{pmatrix}$.

$\det|T_n|^2\det|S|=|S|=|S_{n-1}|d_n$, $d_n>0$.

$\therefore S_n=(T_n^T)^{-1}\begin{pmatrix}S_{n-1}&\\&d_n\end{pmatrix}T_n^{-1}$

$=(T_n^T)^{-1}\begin{pmatrix}T_{n-1}^TT_{n-1}&0\\0&d_n\end{pmatrix}(T_n)^{-1}$

$=(T_n^T)^{-1}\begin{pmatrix}T_{n-1}^T&\\&\sqrt{d_n}\end{pmatrix}\begin{pmatrix}T_{n-1}&\\&\sqrt{d_n}\end{pmatrix}T_n^{-1}$

$=T^TT$,

$T=\begin{pmatrix}T_{n-1}&\\&\sqrt{d_n}\end{pmatrix}\begin{pmatrix}I_{n-1}&-S_{n-1}^{-1}\alpha^T\\&1\end{pmatrix}^{-1}$ 上三角.

T27

$n=1$.　$A>0$, $a_{11}>0$,　A正定.

$|A|=a_{11}$　$\exists(\frac{1}{1}a_{11})^1$ ✓.

$n-1$.　假设 $|A_{n-1}|\leq a_{11}a_{22}\cdots a_{n-1,n-1}$,

~~即~~ $|A_{n-1}|\leq\left(\frac{1}{n-1}\sum_{i=1}^{n-1}a_{ii}a_{ii}\right)^{n-1}$.

$A_{n-1}>0$. $\exists P_{n-1}$, $P_{n-1}^TA_{n-1}P_{n-1}$

对 $A_n=\begin{pmatrix}A_{n-1}&C\\C^T&a_{nn}\end{pmatrix}$,

取 $P_{n_0}=\begin{pmatrix}P_{n-1}&-A_{n-1}^{-1}C\\0&1\end{pmatrix}$,

$P_{n_0}^TA_nP_{n_0}=\begin{pmatrix}P_{n-1}^T&0\\-C^TA_{n-1}^{-1}&1\end{pmatrix}\begin{pmatrix}A_{n-1}&C\\C^T&a_{nn}\end{pmatrix}\begin{pmatrix}P_{n-1}&-A_{n-1}C\\0&1\end{pmatrix}$

$=\begin{pmatrix}P_{n-1}^TA_{n-1}&P_{n-1}^TC\\0&a_{nn}-C^TA_{n-1}^{-1}C\end{pmatrix}\begin{pmatrix}P_{n-1}&-A_{n-1}C\\0&1\end{pmatrix}$

$=\begin{pmatrix}P_{n-1}^TA_{n-1}P_{n-1}&0\\0&a_{nn}-C^TA_{n-1}^{-1}C\end{pmatrix}$