证明　重开

2021年12月15日 星期三　　下午9:07

1. 比较两种变换

① 子空间V中的两组基.

$(\alpha_1 \cdots \alpha_n)$, $(\beta_1 \cdots \beta_n)$, 可相互表示.

设有 $\beta_i = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$.

则有 $\begin{cases} \beta_1 = k_{11}\alpha_1 + \cdots + k_{1n}\alpha_n, \\ \vdots \\ \beta_n = k_{n1}\alpha_1 + \cdots + k_{nn}\alpha_n \end{cases}$

写为 $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$,

也即 $(\beta_1 \cdots \beta_n) = (\alpha_1 \cdots \alpha_n) \begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{nn} \end{pmatrix} = (\alpha_1 \cdots \alpha_n) T$.

对某向量 η,

记为 $\eta = (\alpha_1 \cdots \alpha_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\beta_1 \cdots \beta_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$= (\alpha_1 \cdots \alpha_n) X = (\beta_1 \cdots \beta_n) Y$,

X、Y 为坐标, 则有 $X = TY$.

② 线性变换的两种坐标.

$\mathscr{A}(\alpha_1 \cdots \alpha_n)$ 是对 $(\alpha_1 \cdots \alpha_n)$ 的一种线性变换.

其中 $\mathscr{A}(\alpha_i) = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$
$= (\alpha_1 \cdots \alpha_n)\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$.

则 $\mathscr{A}(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n)\begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{nn} \end{pmatrix}$
$= (\alpha_1 \cdots \alpha_n) T$.

设某一向量 $\eta = (\alpha_1 \cdots \alpha_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

变换后 $\mathscr{A}(\eta) = (\mathscr{A}(\alpha_1 \cdots \alpha_n))\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

则对 $\mathscr{A}(\eta) = (\alpha_1 \cdots \alpha_n) T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\alpha_1 \cdots \alpha_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

即变换前后坐标 $TX = Y$.

2. 可对角化.

3. 度量矩阵.

① 定义.

基 $\alpha_1 \cdots \alpha_n$ 下. $\alpha = (\alpha_1 \cdots \alpha_n) X = \sum_{i=1}^{n} \alpha_i x_i$

$\beta = (\alpha_1 \cdots \alpha_n) Y = \sum_{i=1}^{n} \alpha_i y_i$

$(\alpha, \beta) = (\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{n} \alpha_j y_j)$

$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j (\alpha_i, \alpha_j)$

$= X^T G Y$, 取 $G = ((\alpha_i, \alpha_j))_{n \times n}$.

② 坐标变换.

基 $(\beta_1 \cdots \beta_n)$ 下. $\alpha = (\beta_1 \cdots \beta_n)\tilde{X} = \sum_{i=1}^{n} \beta_i \tilde{x}_i$

$\beta = (\beta_1 \cdots \beta_n)\tilde{Y} = \sum_{i=1}^{n} \beta_i \tilde{y}_i$.

$(\alpha, \beta) = \tilde{X}^T \tilde{G} \tilde{Y}$, $\tilde{G} = ((\beta_i, \beta_j))_{n \times n}$.

变换关系 $(\beta_1 \cdots \beta_n) = (\alpha_1 \cdots \alpha_n) T$.

则对坐标, 如 $\alpha = (\alpha_1 \cdots \alpha_n) X = (\beta_1 \cdots \beta_n) \tilde{X}$,

$\alpha = (\alpha_1 \cdots \alpha_n) T \tilde{X} = (\alpha_1 \cdots \alpha_n) X$, $X = T\tilde{X}$. $X^T = \tilde{X}^T T^T$

$\therefore (\alpha, \beta) = \tilde{X}^T \tilde{G} \tilde{Y} = X^T G Y = (\tilde{X}^T T^T) G (T\tilde{Y})$

$= \tilde{X}^T (T^T G T) \tilde{Y}$, $\therefore \tilde{G} = T^T G T$.

这叫 $\tilde{G} = T^T G T$ 满足相合.

③ 度量矩阵可逆.

证明: 可逆 $\Leftrightarrow$ rank $G = n$ $\Leftrightarrow$ 行秩、列秩为n

$\Leftrightarrow$ 各行/列线性无关.

$G = \begin{pmatrix} (\alpha_1, \alpha_1) & (\alpha_1, \alpha_2) & \cdots & (\alpha_1, \alpha_n) \\ (\alpha_2, \alpha_1) & & \ddots & \vdots \\ \vdots & & & \\ (\alpha_n, \alpha_1) & & & (\alpha_n, \alpha_n) \end{pmatrix}$

验证各个行向量

$k_1((\alpha_1, \alpha_1)\ (\alpha_1, \alpha_2) \cdots (\alpha_1, \alpha_n)) + k_2((\alpha_2, \alpha_1) \cdots (\alpha_2, \alpha_n))$
$+ \cdots + (k_n(\alpha_n, \alpha_1)\ (\alpha_n, \alpha_2) \cdots (\alpha_n, \alpha_n)) = \theta$ 的k的解.

行向量线性
$\Leftrightarrow (k_1(\alpha_1, \alpha_1)\ k_1(\alpha_1, \alpha_2) \cdots k_1(\alpha_1, \alpha_n)) + \cdots$
$+ (k_n(\alpha_n, \alpha_1)\ k_n(\alpha_n, \alpha_2) \cdots k_n(\alpha_n, \alpha_n)) = \theta$.

内积线性
$\Leftrightarrow ((k_1\alpha_1, \alpha_1)\ (k_1\alpha_1, \alpha_2) \cdots (k_1\alpha_1, \alpha_n)) + \cdots$
$+ ((k_n\alpha_n, \alpha_1)\ (k_n\alpha_n, \alpha_2) \cdots (k_n\alpha_n, \alpha_n)) = \theta$.

行向量加
$\Leftrightarrow ((\sum_{i=1}^{n} k_i\alpha_i, \alpha_1)\ (\sum_{i=1}^{n} k_i\alpha_i, \alpha_2) \cdots (\sum_{i=1}^{n} k_i\alpha_i, \alpha_n)) = \theta$.

记 $\alpha = \sum_{i=1}^{n} k_i\alpha_i$, $((\alpha, \alpha_1)\ (\alpha, \alpha_2) \cdots (\alpha, \alpha_n)) = \theta$.

则有 $(\alpha, \alpha_i) = 0$. $k_i(\alpha, \alpha_i) = 0$.

$\therefore \sum_{i=1}^{n} k_i (\alpha, \alpha_i) = 0$, $(\alpha, \sum_{i=1}^{n} k_i\alpha_i) = 0$, $(\alpha, \alpha) = 0$.

内积正定性 $\Rightarrow \alpha = \theta$.

即 $\sum_{i=1}^{n} k_i\alpha_i = \theta$. $\alpha_i \neq \theta$, $\therefore k_i = 0$.

$\therefore$ G 各行向量线性无关, 可逆.

4. 标准正交基.

① 定义.

正交向量组: 两两正交的非零向量.

正交基: 正交向量组成为一组基.

标准正交基: 单位向量构成的正交基.

② 性质: $\alpha_1 \cdots \alpha_n$ 两两正交, 则构成的正交向量组线性无关 (能构成基).

证明: 对方程 $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n = \theta$.

两边同时取与 α_i 的内积,

得到 $(\lambda_1\alpha_1, \alpha_i) + (\lambda_2\alpha_2, \alpha_i) + \cdots = \theta$.

两两正交, 故只有 $\lambda_i(\alpha_i, \alpha_i) = \theta$,

其中 $\alpha_i \neq \theta$, $(\alpha_i, \alpha_i) \neq \theta$, 故 $\lambda_i = 0$,

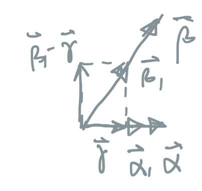
对 $\forall i \in [1, n]$ 成立.

即 $\lambda_1 = \lambda_2 = \cdots = \lambda_i = 0$.

③ 存在性: 取任一组基, 都可构造.

求标准正交基的方法: Schmidt 正交化.

1.



单位化: $\beta_1 = \frac{\beta}{|\beta|}$, $\alpha_1 = \frac{\alpha}{|\alpha|}$.

正交化: 作垂直求成 $\vec{\gamma}$,

$|\gamma| = |\vec{\beta_1}| \cos\langle\vec{\alpha}, \vec{\beta}\rangle$

$= |\vec{\beta_1}| \frac{(\vec{\alpha_1}, \vec{\beta_1})}{|\vec{\alpha_1}||\vec{\beta_1}|}$

$\vec{\gamma} = |\gamma| \frac{\vec{\alpha_1}}{|\vec{\alpha_1}|} = \frac{(\vec{\alpha_1}, \vec{\beta_1})}{|\vec{\alpha_1}||\vec{\alpha_1}|=1} \vec{\alpha_1}$.

$\vec{\beta_1} - \vec{\gamma} = \vec{\beta_1} - (\vec{\alpha_1}, \vec{\beta_1})\vec{\alpha_1}$.

2. 取基 $\alpha_1 \cdots \alpha_n$, 求正交基.

① 对 $\vec{\alpha_1}$, 取单位 $\vec{e_1} = \frac{\vec{\alpha_1}}{|\alpha_1|}$.

② 对 $\vec{e_1}$、$\vec{\alpha_2}$ 取垂直向量 $\vec{\beta_2}$:

$\vec{\beta_2} = \vec{\alpha_2} - (\vec{e_1}, \vec{\alpha_2})\vec{e_1}$.

[可验证 $(\vec{\beta_2}, \vec{e_1}) = (\vec{\alpha_2} - (\vec{e_1}, \vec{\alpha_2})\vec{e_1}, \vec{e_1})$
$= (\vec{\alpha_2} - \frac{1}{|\alpha_1|}(\vec{\alpha_1}, \vec{\alpha_2})\frac{\vec{\alpha_1}}{|\alpha_1|}, \frac{\vec{\alpha_1}}{|\alpha_1|})$
$= (\vec{\alpha_2} - \frac{(\vec{\alpha_1}, \vec{\alpha_2})}{|\alpha_1|^2}\vec{\alpha_1}, \frac{\vec{\alpha_1}}{|\alpha_1|}) = 0$,
即 $\vec{\beta_2}$ 与 $\vec{e_1}$ 正交.]

及时取单位 $\vec{e_2} = \frac{\vec{\beta_2}}{|\beta_2|}$.

③ 对 $\vec{e_1}$、$\vec{e_2}$、$\vec{\alpha_3}$ 取垂直:

$\vec{\beta_3} = \vec{\alpha_3} - (\vec{e_1}, \vec{\alpha_3})\vec{e_1} - (\vec{e_2}, \vec{\alpha_3})\vec{e_2}$.

[可验证 $(\vec{\beta_3}, \vec{e_1}) = (\vec{\alpha_3} - (\vec{e_1}, \vec{\alpha_3})\vec{e_1} - (\vec{e_2}, \vec{\alpha_3})\vec{e_2}, \vec{e_1})$
$= (\vec{\alpha_3} - (\vec{e_1}, \vec{\alpha_3})\vec{e_1}, \vec{e_1}) = 0$,
同理 $(\vec{\beta_3}, \vec{e_2}) = 0$.
即 $\vec{\beta_3}$ 与 $\vec{e_1}$、$\vec{e_2}$ 都正交.]

及时取单位 $\vec{e_3} = \frac{\vec{\beta_3}}{|\beta_3|}$.

……

ⓝ 对 $\vec{e_1} \cdots \vec{e_{n-1}}$、$\vec{\alpha_n}$ 取垂直

$\vec{\beta_n} = \vec{\alpha_n} - \sum_{i=1}^{n-1}(\vec{e_i}, \vec{\alpha_n})\vec{e_i}$. $\vec{e_n} = \frac{\vec{\beta_n}}{|\beta_n|}$

$\therefore \vec{e_1} \cdots \vec{e_n}$ 为标准正交基.

5. 正交变换.

① 几些等价命题.

② 正交阵性质.

6. 对称变换.

① 定义.

② 对应矩阵.

③ 可对角化.