例题、错题の重开

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一、求解线性方程组

1. $\begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -6 & 4 & -1 \\ 3 & 2 & p & 7 & -1 \\ 1 & -1 & -6 & -1 & q \end{pmatrix}$ 讨论 p, q 对解的影响.

解: $\begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -6 & 4 & -1 \\ 3 & 2 & p & 7 & -1 \\ 1 & -1 & -6 & -1 & q \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \\ 0 & -1 & p+6 & -2 & -1 \\ 0 & -2 & -4 & -4 & q \end{pmatrix}$

$\to \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & p+8 & 0 & 0 \\ 0 & 0 & 0 & 0 & q+2 \end{pmatrix}$

$p=-8$: $q=-2 \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \end{pmatrix}$, $x_2 = -2x_3 - 2x_4 + 1$, $x_1 = -x_2 + 2x_3 + 3x_4 = x_4 + 1$.

$q \neq -2 \Rightarrow$ 无解.

$p \neq -8 \Rightarrow x_3 = 0$.

$q=-2 \Rightarrow x_4 = t$, $x_2 = -2t+1$, $x_1 = t+1$.

$q \neq -2 \Rightarrow$ 无解.

分类: $q \neq -2$, 无解.

$q=-2$: $p=-8$. 令 $x_3 = t_1$, $x_4 = t_2$, $x_2 = 1 - 2t_1 - 2t_2$, $x_1 = -1 - t_2 + 4t_1$.

$p \neq -8$. $x_3 = 0$. 令 $x_4 = t$, $x_2 = 1-2t$, $x_1 = -t-1$.

从 $d+1=0$ 开始分类.

例. 设复矩阵满足 $tr(A\bar{A}^T)=0$, 证明: $A=0$.

设 $A=(a_{ij})_{m\times n}$. $\bar{A}=(\bar{a}_{ij})_{m\times n}$. $\bar{A}^T=(\bar{a}_{ji})_{n\times m}$.

$A\bar{A}^T = (a_{ij}\times \bar{a}_{jj})_{m\times m}$

$tr(A\bar{A}^T) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\bar{a}_{ij} = 0$,

$\therefore a_{ij}\bar{a}_{ij} = |a_{ij}|^2 = 0$.

$\therefore a_{ij}=0$, $A=0$.

例. A. B. I 为 n 阶方阵, $BA=0$. $M = \begin{pmatrix} I & A \\ B & I \end{pmatrix}$ 求逆.

$\begin{pmatrix} I & A & I & 0 \\ B & I & 0 & I \end{pmatrix} \to \begin{pmatrix} I-BA & 0 & I & -A \\ B & I & 0 & I \end{pmatrix}$

$= \begin{pmatrix} I & 0 & I & -A \\ B & I & 0 & I \end{pmatrix} \to \begin{pmatrix} I & 0 & I & -A \\ 0 & I & -B & I+AB \end{pmatrix}$

$\therefore M^{-1} = \begin{pmatrix} I & -A \\ -B & I+AB \end{pmatrix}$ $\begin{pmatrix} I+AB & -A \\ -B & I \end{pmatrix}$.

相抵 → 秩一样, 都可变为 I_r

例. $\det \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \vdots \\ & & & A_{kk} \end{pmatrix} = \det A_{11} \cdots \det A_{kk}$

$k=2$. $\det \begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{r,r+1} & \cdots & a_{rn} \\ & & & a_{r+1,r+1} & \cdots & a_{r+1,n} \\ & 0 & & \vdots & & \vdots \\ & & & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix}$

$= \sum_{\substack{1\le j_1 \cdots j_r \le r \\ r+1 \le j_{r+1} \cdots j_n \le n}} (-1)^{\tau(j_1 \cdots j_r)+\tau(j_{r+1}\cdots j_n)} a_{1j_1} \cdots a_{nj_n}$

$= \sum_{1\le j_1 \cdots j_r \le r} (-1)^{\tau(j_1\cdots j_r)} a_{1j_1} \cdots a_{rj_r} + \sum_{r+1\le j_{r+1} \cdots j_n \le n} (-1)^{\tau(j_{r+1}\cdots j_n)} a_{r+1,j_{r+1}} \cdots a_{nj_n}$

$= \det A_{11} \det A_{22}$

$k\ge 3$. $A \to \begin{pmatrix} A_{11} & B_{12} \\ 0 & B \end{pmatrix}$ 递推

二、矩阵

例. 求斐波那契数列的通项公式.

$F_n = F_{n-1} + F_{n-2} \Rightarrow F_n = (1 \ 1)\begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$

保证形式相同, $\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$

$F_n + F_{n-1} = 2F_{n-1} + F_{n-2}$.

$\therefore \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2\begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$

$= \cdots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore F_n = (1 \ 0)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

例. 证明: (1) $ad \neq bc$ 时, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(2) $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$, $\vec{c}=(c_1,c_2,c_3)$. $(\vec{a}\times\vec{b})\cdot\vec{c} \neq 0$ 时,

有 $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}^{-1} = \frac{1}{(\vec{a}\times\vec{b})\cdot\vec{c}}\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$

其中 $\vec{u}=\vec{b}\times\vec{c}$, $\vec{v}=\vec{c}\times\vec{a}$, $\vec{w}=\vec{a}\times\vec{b}$.

(1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix}$

$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)I$.

$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(2) $\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = (\vec{u}\ \vec{v}\ \vec{w})\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$

$= \vec{u}\times\vec{a} + \vec{v}\times\vec{b} + \vec{w}\times\vec{c}$

$= \vec{b}\times\vec{c}\times\vec{a} + \vec{c}\times\vec{a}\times\vec{b} + \vec{a}\times\vec{b}\times\vec{c}$.

$= \vec{a}\times\vec{b}\times\vec{c}$.

$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} = \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}(\vec{u}\ \vec{v}\ \vec{w})$

$= \begin{pmatrix} \vec{a}\cdot\vec{u} & \vec{a}\cdot\vec{v} & \vec{a}\cdot\vec{w} \\ \vec{b}\cdot\vec{u} & \vec{b}\cdot\vec{v} & \vec{b}\cdot\vec{w} \\ \vec{c}\cdot\vec{u} & \vec{c}\cdot\vec{v} & \vec{c}\cdot\vec{w} \end{pmatrix}$

例. $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, 求 A^n

例. 计算 n 阶方阵的秩.

例. 计算 A 的秩.

A 的右上角 n-1 阶子式非零, $\therefore rank A \ge n-1$.

例. 计算 A 的秩.

1. 已知 A^*, 问 A 是否可逆. 如果可逆, 求 A^{-1}.

$\det(A^*) = 8$, A^* 可逆.

若 A 不可逆, 则 $\det A = 0$, 即 $AA^* = \det A = 0$.

$\therefore A = 0$. $\therefore A^* = 0$, 与条件矛盾. $\therefore A$ 可逆.

$\det(A^*) = (\det A)^{n-1} = 8$, $\det A = 2$.

$\therefore A^{-1} = \frac{1}{\det A} A^*$

2. 求

△12.7

例: 设复矩阵 A 满足 $tr(A\bar{A}^T)=0$, 证明 $A=0$.

$A=(a_{ij})_{m\times n}$, $A\bar{A}^T = (b_{ij})_{m\times m}$. 乘后结果.

$0 = tr(A\bar{A}^T) = \sum_{i=1}^{m} b_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\bar{a}_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2$.

$\therefore$ 所有 $a_{ij}=0$, $A=0$.

例: 上三角阵相乘后还是上三角阵.

上三角阵 $A_{n\times n}=(a_{ij})_{n\times n}$, 其中 $a_{ij}=0$, $i>j$.

设 $C = A\cdot B = (c_{ij})_{n\times n} = (\sum_{k=1}^{n} a_{ik}b_{kj})_{n\times n}$.

$i\le j$, $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

$i>j$, $i>k$, $a_{ik}=0$.

$i\le k$, 此时 $k\ge i>j$, $b_{kj}=0$.

$\therefore c_{ij}=0$. $\therefore$ 是上三角阵.

例. 每个秩为 r 的矩阵都可以写成 r 个秩为 1 的矩阵之和.

设 $A = P\,diag(I_r, 0)Q$. 其中 P. Q 可逆.

$\therefore A = PE_{11}Q + PE_{22}Q + \cdots + PE_{rr}Q$.

例. 若 $m\times n$ 矩阵 A 列满秩, 则 A 是某个可逆方阵的前 n 列.

列满秩 $\Rightarrow m \ge n = rank(A)$.

设 $A = P\begin{pmatrix} I_n \\ 0 \end{pmatrix}Q$, 其中 P. Q 可逆.

则 $A = P\begin{pmatrix} Q \\ 0 \end{pmatrix}$, 是 m 阶可逆方阵 $P\,diag(Q, I_{m-n})$ 的前 n 列.

例. 对任意 $A_{m\times n}$, $B_{n\times p}$, 都有 $rank(AB) \le \min(rank(A), rank(B))$.

$A = P_1\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}Q_1$, $B = P_2\begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}Q_2$.

$AB = P_1\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}Q_1P_2\begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}Q_2 = P_1\begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix}Q_2$.

将 Q_1P_2 分块成矩阵 $(R_{ij})_{2\times 2}$, 其中 R_{11} 是 $r\times s$ 矩阵.

$\therefore rank(AB) = rank\left(P_1\begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix}Q_2\right) \le \min(r, s)$.

例. 设 A 为 n 阶方阵, I 为同阶单位矩阵, 且 $A^2=A$. 证明: $rank(A) + rank(I-A) = n$.

$rank(A) + rank(I-A) = rank\begin{pmatrix} A & 0 \\ 0 & I-A \end{pmatrix}$.

$\begin{pmatrix} A & 0 \\ 0 & I-A \end{pmatrix} \to \begin{pmatrix} A & 0 \\ A & I-A \end{pmatrix} \to \begin{pmatrix} A & A \\ A & I \end{pmatrix} \to \begin{pmatrix} A-A^2 & 0 \\ A & I \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, $rank = n$.

1. 证明: 任一方阵都可以表示为一个对称矩阵和一个反对称矩阵之和.

$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$.

2. 设 A. B 都是 n 阶对称方阵, 且 $AB=BA$. 证明: AB 也是对称方阵.

对称 $\Rightarrow A=A^T$, $B=B^T$.

$(AB)^T = B^TA^T = BA = AB$. $\therefore$ 对称.

3. 证明: 与任意 n 阶方阵都乘法可交换的方阵一定是数量矩阵.

设 A 与任意 n 阶方阵乘法可交换.

取 E_{ij}, $AE_{ij} = E_{ij}A$, $i \neq j$.

则可得 $a_{ii} = a_{jj}$, $a_{ik}=0$, $a_{kj}=0$.

4. 设 A^* 表示 n 阶方阵 A 的伴随方阵.

证明: (1) $(\lambda A)^* = \lambda^{n-1}A^*$.
(2) $(AB)^* = B^*A^*$.
(3) $\det(A^*) = (\det(A))^{n-1}$

(1) $\lambda = 0$, $(\lambda A)^* = 0 = \lambda^{n-1}A^*$.

$\lambda \neq 0$, $(\lambda A)^* = |\lambda A|(\lambda A)^{-1} = \lambda^n|A|\lambda^{-1}A^{-1} = \lambda^{n-1}A^*$.

(2) A.B 都可逆, $(AB)^* = |AB|(AB)^{-1}$

$= (|B|B^{-1})(|A|A^{-1}) = B^*A^*$.

(3) $A^* = \det(A)A^{-1}$, $\det(A^*) = [\det A]^n(\det A)^{-1} = (\det A)^{n-1}$

A.B 至少有一个不可逆,

令 $A_t = A + tE$, $B_t = B + tE$,

$|A^*| = |A|^n|A|^{-1}$, $(A^*)^{-1} = |A|^{-1}A \Rightarrow (A^*)^* = |A^*|(A^*)^{-1} = |A|^{n-2}A$.

则 $\exists \delta > 0$, 当 $t\in(0,\delta)$ 时, A_t. B_t 可逆.

$\therefore$ 有 $(A_tB_t)^* = B_t^*A_t^*$, 为含 t 的多项式.

$\therefore t\to 0$ 时, 多项式仍成立. $(AB)^* = B^*A^*$.

5. 设 $A_{n\times n}$, 证明: $rank(A^*) = \begin{cases} n, & rank(A) = n \\ 1, & rank(A) = n-1 \\ 0, & rank(A) \le n-2 \end{cases}$.

$rank\,A = n$, $\therefore \det A \neq 0$, $\therefore \det A^* = (\det(A))^{n-1} \neq 0$.

$\therefore rank(A^*) = n$.

$rank\,A \neq n$, 有 $AA^* = \det(A) = 0$.

A^* 的每一列都是齐次方程组 $Ax=0$ 的解.

$\therefore rank(A) + rank(A^*) \le n$.

$rank\,A = n-1$, 即 A 存在 n-1 阶子式不为零,

$\therefore A^*$ 不是零矩阵, $rank\,A^* \ge 1$.

又 $rank(A^*) \le n - rank(A) = 1$, $\therefore rank\,A^* = 1$.

$rank\,A \le n-2$, 即 A 不存在不为零的 n-1 阶子式.

$\therefore A^*$ 是零矩阵, $rank\,A^* = 0$.

三、行列式

例. 证明 $\det(\alpha+\beta, \beta+\gamma, \gamma+\alpha) = 2\det(\alpha, \beta, \gamma)$.

$\det(\alpha+\beta, \beta+\gamma, \gamma+\alpha)$

$= \det(\alpha, \beta+\gamma, \gamma+\alpha) + \det(\beta, \beta+\gamma, \gamma+\alpha)$

$= \det(\alpha, \beta+\gamma, \gamma) + \det(\beta, \gamma, \gamma+\alpha)$

$= \det(\alpha, \beta, \gamma) + \det(\beta, \gamma, \alpha)$

$= 2\det(\alpha, \beta, \gamma)$.

例. 准上三角形方阵 $A = (A_{ij})_{k\times k}$ 的所有对角块 A_{ii} 都是方阵. 求证:

$\det(A) = \det(A_{11})\det(A_{22})\cdots\det(A_{kk})$.

$k=1$, $\det(A) = \det|A_{11}| = \det(A_{11})$

$k=2$, $\det(A) = \det\begin{vmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{vmatrix} = \det$

按行展开.

$\det A = \sum_{\substack{1\le j_1\cdots j_r \le r \\ r+1 \le j_{r+1}\cdots j_n \le n}} (-1)^{\tau(j_1\cdots j_r)+\tau(j_{r+1}\cdots j_n)} \prod_{i=1}^{n} a_{ij_i}$

$= \sum_{1\le j_1\cdots j_r\le r} (-1)^{\tau(j_1\cdots j_r)} \prod_{i=1}^{n} a_{ij_i}$

$= \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}\begin{vmatrix} a_{r+1,r+1} & \cdots & a_{r+1,n} \\ \vdots & & \vdots \\ a_{n,r+1} & \cdots & a_{nn} \end{vmatrix}$

$k\ge 3$, A 分块成 $\begin{pmatrix} A_{11} & A_{12} \\ 0 & B \end{pmatrix}$.

$\det(A) = \det(A_{11})\det(B) = \cdots$

例. 设 A 是一个 $m\times n$ 矩阵, B 是一个 $n\times m$ 矩阵, $m>n$. 证明: $\det(AB) = 0$.

$m>n$, 在 A 的右边、B 的下边添加 m-n 行零向量形成方阵 $\tilde{A}=(A\ 0)$, $\tilde{B}=\begin{pmatrix} B \\ 0 \end{pmatrix}$.

$\therefore \tilde{A}\tilde{B} = AB$.

$\det(AB) = \det(\tilde{A}\tilde{B}) = \det(\tilde{A})\det(\tilde{B}) = 0$.

1. 设 A 是奇数阶反对称复方阵, 证明 $\det A = 0$.

反对称 $\Rightarrow A + A^T = 0$, $A = -A^T$.

$\det A = -\det(A^T)$.

又对于方阵, $\det A = \det A^T$.

$\therefore \det A = \det(A^T) = 0$.

2. 设 A 是偶数阶反对称复方阵, 证明其所有元素的代数余子式 A_{ij} 之和为 0.

扩充一行一列. $\begin{vmatrix} 0 & 1 & \cdots & 1 \\ -1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ -1 & a_{n1} & \cdots & a_{nn} \end{vmatrix}$ ⇒ 奇数阶反对称.

$\therefore$ 扩充之后的 $\det(\tilde{A}) = 0$.

将 $|B|$ 按第 1 行展开有

$0 = |B| = \sum_{j=0}^{n} b_{1,j+1}B_{1,j+1} = \sum_{j=1}^{n} (-1)^{1+j+1}N_{1,j+1}$.

其中 $N_{1,j+1}$ 是 B 第 $(1, j+1)$ 元素的余子式, 由 B 删去第 1 行、除上一列得到.

又 $N_{1,j+1} = \begin{vmatrix} -1 & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots \\ \vdots \\ -1 & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots \end{vmatrix}$ 按第 1 列展开.

$N_{1,j+1} = \sum_{i=1}^{n} (-1)(-1)^{i+1}(N_{1,j+1})_{i1}$.

其中 $(N_{1,j+1})_{i1}$ 由 A 删去第 i 行第 j 列得到, 即为余子式 M_{ij}. $\therefore N_{1,j+1} = \sum_{i=1}^{n}(-1)^i M_{ij}$.

$\therefore 0 = |B| = \sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^j(-1)^i M_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}$.

又 $\Delta = \begin{vmatrix} b_0 & c_1 & \cdots & c_n \\ c_1 & a_{11} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ c_n & a_{n1} & \cdots & a_{nn} \end{vmatrix} = b_0|A| - \sum_{1\le i,j\le n} b_ic_jA_{ij}$.

3. 设 n 阶方阵 A 的每行、每列元素之和都是 0, 证明: A^* 的所有元素相等.

设 $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

$= \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = 0$.

$A_{i,j+1} = (-1)^{i+j+1}M_{i,j+1} = (-1)^{i+j+1}(-1)M_{ij} = A_{ij}$

$A_{i+1,j} = (-1)^{i+j+1}M_{i+1,j} = (-1)^{i+j+1}(-1)M_{ij} = A_{ij}$.

所有元素和为 0. 移动一下, 为负.

4. 设 A 是 n 阶可逆方阵, $\alpha = (a_1, \cdots a_n)^T$.

证明: $\det(A - \alpha\alpha^T) = (1 - \alpha^TA^{-1}\alpha)\det(A)$.

$\det(A - \alpha\alpha^T) = \det(A(I_n - A^{-1}\alpha\alpha^T))$

$= \det(A)\det(I_n - A^{-1}\alpha\alpha^T)$

公式: $\det(I_n - A_{n\times m}B_{m\times n}) = \det(I_m - B_{m\times n}A_{n\times m})$

$\Rightarrow \det(I_n - A^{-1}_{n\times n}\alpha_{n\times 1}\alpha^T_{1\times n}) = \det(I_1 - \alpha^T_{1\times n}A^{-1}_{n\times n}\alpha_{n\times 1})$.

$= \det(A)\det(I_1 - \alpha^TA^{-1}\alpha)$

$= (1 - \alpha^TA^{-1}\alpha)\det(A)$.

5. 设 $A \in F^{n\times m}$, $B \in F^{m\times n}$, 求证:

$\lambda^m|\lambda I_n - AB| = \lambda^n|\lambda I_m - BA|$