例题

2022年5月16日 星期一 上午8:02

一、静电能

(1) 带电体在外场中.

例. 边长为 a 的正六边形

(1) 系统静电能.

(2) 将两个点电荷移至无穷远, 外力做的功.

例. 无限长, 相距 a. 计算一个离子与其他的相互作用能.

例. 均匀带电 Q 球面 R 的静电能.

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例. 聚拢电荷为球体外界做功.

(2) 带电体系

例. 电偶极子的能量. 外场中的电势能: 电偶极子相互作用能:

例. 球形电容器的静电能.

例. 球形电容.

例. 平行板电容器.

(3) 能量密度.

例. 球形电容器的静电能.

例. 把电荷为 q 的粒子从无限远处移到半径为 R, 厚度为 t 的空心导体球壳中心, 此过程要做多少功?

例. 平行板电容器极板间充满 ε 的介质, 面积 S 间距 d. 则在极板上带 $+Q_0$ 和 $-Q_0$ 的电荷, 外界需要抵抗静电力做多少功?

例. 边长为 a 的正六边形

(1) 系统静电能.

(2) 将两个点电荷移至无穷远, 外力做的功.

对每个 Q 受其它 Q 的电势:

$$\frac{2\sqrt{3}}{4\sqrt{3}}=\frac{2}{\sqrt{3}}$$

$$\varphi_-=2\times\frac{Q}{4\pi\varepsilon_0 a}-2\times\frac{Q}{4\pi\varepsilon_0\sqrt{3}a}+\frac{Q}{4\pi\varepsilon_0 2a}$$

$$=\frac{Q}{4\pi\varepsilon_0 a}\left(2-\frac{2}{\sqrt{3}}+\frac{1}{2}\right)=\frac{Q}{4\pi\varepsilon_0 a}\times\left(\frac{5}{2}-\frac{2}{\sqrt{3}}\right)$$

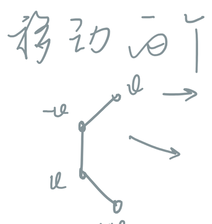
$$\varphi_+=-\varphi_-$$

$$W=\frac{1}{2}\left(3Q\varphi_++3(-Q)\varphi_-\right)$$

$$=\frac{3}{2}\left(Q\times\left(-\frac{Q(5\sqrt{3}-2)}{4\sqrt{3}\pi\varepsilon_0 a}\right)+(-Q)\left(\frac{Q(5\sqrt{3}-2)}{4\sqrt{3}\pi\varepsilon_0 a}\right)\right)$$

$$=-\frac{3}{2}\times2\frac{Q^2(5\sqrt{3}-2)}{4\sqrt{3}\pi\varepsilon_0 a}=\frac{3Q^2}{4\pi\varepsilon_0 a}\left(\frac{2}{\sqrt{3}}-\frac{5}{2}\right)<0.$$

移动两个后



$$\varphi_{+1}=-\frac{Q}{4\pi\varepsilon_0 a}+\frac{Q}{4\pi\varepsilon_0\sqrt{3}a}-\frac{Q}{4\pi\varepsilon_0 2a}=\frac{Q}{4\pi\varepsilon_0 a}\left(\frac{1}{\sqrt{3}}-\frac{3}{2}\right)$$

$$\varphi_{-2}=2\times\frac{Q}{4\pi\varepsilon_0 a}-\frac{Q}{4\pi\varepsilon_0\sqrt{3}a}=\frac{Q}{4\pi\varepsilon_0 a}\left(2-\frac{1}{\sqrt{3}}\right).$$

$$W_1=\frac{1}{2}\left(Q\times\varphi_{+1}-Q\varphi_{-2}+Q\varphi_{+2}-Q\varphi_{-1}\right)$$

$$=\frac{1}{2}\left(\frac{Q^2}{4\pi\varepsilon_0 a}\left(\frac{1}{\sqrt{3}}-\frac{3}{2}\right)-\frac{Q^2}{4\pi\varepsilon_0 a}\left(2-\frac{1}{\sqrt{3}}\right)\right.$$

$$\left.+\frac{Q^2}{4\pi\varepsilon_0 a}\left(\frac{1}{\sqrt{3}}-2\right)-\frac{Q^2}{4\pi\varepsilon_0 a}\left(\frac{3}{2}-\frac{1}{\sqrt{3}}\right)\right)$$

$$=\frac{Q^2}{4\pi\varepsilon_0 a}\left(\frac{1}{\sqrt{3}}-\frac{3}{2}+\frac{1}{\sqrt{3}}-2\right)=\frac{Q^2}{4\pi\varepsilon_0 a}\left(\frac{2}{\sqrt{3}}-\frac{7}{2}\right)$$

$$W_2=\frac{Q^2}{4\pi\varepsilon_0 a}.\quad\therefore W'=\frac{Q^2}{4\pi\varepsilon_0 a}\left(\frac{2}{\sqrt{3}}-\frac{5}{2}\right).$$

$$\text{外力做功}=W'-W=\frac{Q^2}{4\pi\varepsilon_0 a}\left(5-\frac{4}{\sqrt{3}}\right)>0.$$

例. 无限长, 相距 a. 计算一个离子与其他的相互作用能.

$$\varphi_+=-2\frac{q}{4\pi\varepsilon_0 a}+2\frac{q}{4\pi\varepsilon_0 2a}-\cdots=\frac{q}{2\pi\varepsilon_0 a}\left(-1+\frac{1}{2}-\frac{1}{3}+\cdots\right)$$

$$=\frac{q}{2\pi\varepsilon_0 a}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}=-\frac{q}{2\pi\varepsilon_0 a}\ln 2.$$

例. 均匀带电 Q 球面 R 的静电能.

球面电势 $\varphi=\frac{Q}{4\pi\varepsilon_0 a}$

静电能 $W=\frac{1}{2}\int dq\,\varphi=\frac{1}{2}Q\frac{Q}{4\pi\varepsilon_0 a}=\frac{Q^2}{8\pi\varepsilon_0 a}$

例. 均匀带电 Q 球体 R 的静电能.

$$\vec{E}=\begin{cases}\frac{1}{4\pi\varepsilon_0 r^2}\frac{r^3}{R^3}Q, & r<R\\ \frac{Q}{4\pi\varepsilon_0 r^2}, & r\geq R\end{cases}$$

$$U=\begin{cases}\frac{Q}{4\pi\varepsilon_0 R}+\frac{Q R^2}{24\pi\varepsilon_0 R^3}\ldots, & r<R\\ \frac{Q}{4\pi\varepsilon_0 R}, & r\geq R\end{cases}\qquad \rho=\frac{Q}{\frac{4}{3}\pi R^3}\qquad \frac{\rho R^2}{6\varepsilon_0}\left(1-\frac{r^2}{R^2}\right)$$

球内静电能 $W=\frac{1}{2}\int_0^R\frac{\rho R^2}{6\varepsilon_0}\left(3-\frac{r^2}{R^2}\right)\rho\,4\pi r^2dr$

$$=\frac{4\pi\rho^2R^5}{15\varepsilon_0}=\frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 a}.$$

例. 聚拢电荷为球体外界做功.

假设已经聚成半径 r 的球, $r<a$.

球内电量 $Q_r=\frac{r^3}{a^3}Q$ → 均匀分布, 处处

球面电势 $\varphi_s=\frac{Q_r}{4\pi\varepsilon_0 r}$

则再移为半径 $r+dr$ 的球时: dr 中的 dQ.

外界做功 $\varphi_s dQ=\frac{1}{4\pi\varepsilon_0 r}\cdot\frac{r^3}{a^3}Q\times\frac{4\pi r^2dr}{\frac{4}{3}\pi a^3}Q$

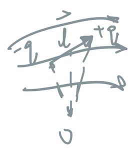
$$=\frac{3Q^2}{4\pi\varepsilon_0 a^6}r^4dr.$$

∴移动总做功 $W=\int_0^a\frac{3Q^2}{4\pi\varepsilon_0 a^6}r^4dr$

$$=\frac{3Q^2}{4\pi\varepsilon_0 a^6}\cdot\frac{1}{5}a^5=\frac{3Q^2}{20\pi\varepsilon_0 a}.$$

例. 电偶极子的能量.

外场中的电势能:



$$U=q\varphi_+-q\varphi_-\qquad \varphi\to\text{电势函数}$$

$$=q\underline{\vec{l}}\cdot\underline{\nabla\varphi_e(\vec{r})}\qquad \vec{l}\to\text{方向向量}$$

(取负)

$$=-\vec{p}\cdot\vec{E}_0.$$

电偶极子相互作用能:

$$W_{12}=-\vec{p}_1\cdot\vec{E}_2=-\vec{p}_1\cdot\frac{3(\vec{p}_2\cdot\hat{r}_{12})\hat{r}_{12}-\vec{p}_2}{4\pi\varepsilon_0 r_{12}^3}$$

$$=\frac{\vec{p}_1\cdot\vec{p}_2-3(\vec{p}_2\cdot\hat{r}_{12})(\vec{p}_1\cdot\hat{r}_{12})}{4\pi\varepsilon_0 r_{12}^3}$$

$$=-\frac{1}{2}(\vec{p}_1\cdot\vec{E}_2+\vec{p}_2\cdot\vec{E}_1)$$

$$W=-\frac{1}{2}\sum_{i=1}^{N}\vec{p}_i\cdot\vec{E}_i.$$

$$W_{12}=\frac{\vec{p}_1\cdot\vec{p}_2-3(\vec{p}_2\cdot\hat{r}_{12})(\vec{p}_1\cdot\hat{r}_{12})}{4\pi\varepsilon_0 r_{12}^3}$$

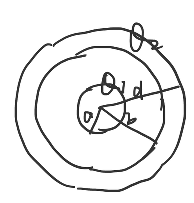
⇒ $\vec{p}_1$ → $\vec{p}_2$ →: $W_{12}=-\frac{2p_1p_2}{4\pi\varepsilon_0 r_{12}^3}<0.$

$\vec{p}_1$ → $\vec{p}_1$ ←: $W_{12}=+\frac{2p_1p_2}{4\pi\varepsilon_0 r_{12}^3}>0.$

$\vec{p}_1$ ↑ $\vec{p}_2$ ↓: $W_{12}=+\frac{p_1p_2}{4\pi\varepsilon_0 r_{12}^3}>0.$

$\vec{p}_1$ ↑ $\vec{p}_2$ ↑: $W_{12}=-\frac{p_1p_2}{4\pi\varepsilon_0 r_{12}^3}<0.$

例. 球形电容器的静电能.



内球表面带 Q_1.

外球内表面感应 $-Q_1$,

外表面带电 Q_2+Q_1.

$$W=W_a+W_b+W_d+W_{ab}+W_{bd}+W_{ad}$$

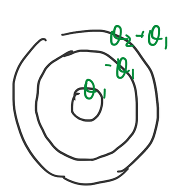
$$=\frac{1}{2}\left(\frac{Q_1^2}{4\pi\varepsilon_0 a}+\frac{(-Q_1)^2}{4\pi\varepsilon_0 b}+\frac{(Q_2+Q_1)^2}{4\pi\varepsilon_0 d}\right)$$

$$+\frac{Q_1(-Q_1)}{4\pi\varepsilon_0 b}+\frac{(-Q_1)(Q_2+Q_1)}{4\pi\varepsilon_0 d}+\frac{Q_1(Q_2+Q_1)}{4\pi\varepsilon_0 d}$$

$$=\frac{1}{2}Q_1^2\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{2}\frac{(Q_1+Q_2)^2}{4\pi\varepsilon_0 d}$$

$Q_1=Q=-Q_2$ 时, $W=\frac{Q^2}{2C}=\frac{1}{2}QV=\frac{1}{2}CV^2.$

例. 球形电容.



$$W=\frac{1}{2}Q_1\varphi_1+\frac{1}{2}Q_2\varphi_2$$

$$=\underbrace{\frac{1}{2}Q_1^2\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)}_{\text{电荷量为}Q_1\text{部分的储能}}+\underbrace{\frac{1}{2}\frac{(Q_1+Q_2)^2}{4\pi\varepsilon_0 d}}_{Q_1+Q_2\text{部分}}$$

$$W=\frac{1}{2}Q_1\varphi_1+\frac{1}{2}Q_2\varphi_2$$

$$=\frac{1}{2}Q_1(\varphi_1-\varphi_2)+\frac{1}{2}(Q_1+Q_2)\varphi_2$$

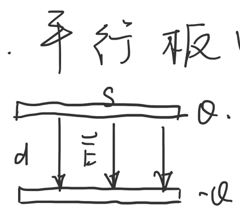
$$=\frac{1}{2}Q_1V_{12}+\frac{1}{2}(Q_1+Q_2)\varphi_2$$

$$\varphi_1=\frac{Q_1+Q_2}{4\pi\varepsilon_0 d}+\frac{Q_1}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right),\quad\varphi_2=\frac{Q_1+Q_2}{4\pi\varepsilon_0 d}.$$

$$E=\begin{cases}\frac{Q_1}{4\pi r^2\varepsilon_0}, & a<r<b\\ 0, & b<r<d\\ \frac{Q_1+Q_2}{4\pi r^2\varepsilon_0}, & r>d\end{cases}$$

$$\varphi=\begin{cases}\frac{Q_1}{4\pi\varepsilon_0 r}+\frac{Q_1+Q_2}{4\pi\varepsilon_0 d}, & a<r<b\\ \frac{Q_1+Q_2}{4\pi\varepsilon_0 d}, & b<r<d.\\ \frac{Q_1+Q_2}{4\pi\varepsilon_0 r}, & r>d.\end{cases}$$

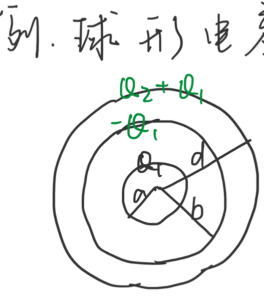
例. 平行板电容器.



$$W=\frac{1}{2}Q\varphi_1+\frac{1}{2}(-Q)\varphi_2=\frac{1}{2}QV$$

$$=\frac{1}{2}\varepsilon_0 ES\cdot Ed=\frac{1}{2}\varepsilon_0E^2Sd.$$

2. 例. 球形电容器的静电能.



$$\vec{E}_{内}=\frac{Q_1}{4\pi\varepsilon_0 r^2}\hat{r},\quad\vec{E}_{外}=\frac{Q_1+Q_2}{4\pi\varepsilon_0 r^2}\hat{r}.$$

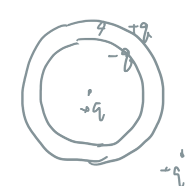
静电能密度 $w_{内}=\frac{Q_1^2}{32\pi^2\varepsilon_0}\times\frac{1}{r^4}$

$$w_{外}=\frac{(Q_1+Q_2)^2}{32\pi^2\varepsilon_0}\times\frac{1}{r^4}.$$

$$W_{内}=\iiint w_{内}dV=\frac{Q_1^2}{32\pi^2\varepsilon_0}\int_a^b\frac{1}{r^4}4\pi r^2dr=\frac{1}{2}\frac{Q_1^2}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)=\frac{Q_1^2}{2C}$$

$$W_{外}=\iiint w_{外}dV=\frac{(Q_1+Q_2)^2}{32\pi^2\varepsilon_0}\int_d^{\infty}\frac{1}{r^4}4\pi r^2dr=\frac{1}{2}\frac{(Q_1+Q_2)^2}{4\pi\varepsilon_0}\frac{1}{d}.$$

例. 把电荷为 q 的粒子从无限远处移到半径为 R, 厚度为 t 的空心导体球壳中心, 此过程要做多少功?



$$E_0=\frac{q}{4\pi\varepsilon_0 r^2}$$

$$V=\frac{q}{4\pi\varepsilon_0 R}$$

原: 无限远激发电场 $E=\frac{q}{4\pi\varepsilon_0 r^2}$. 存全空间

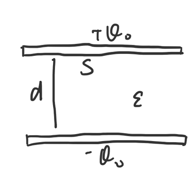
后: 移至导体中心, 导体内 E=0.

$$W_1=\frac{1}{2}\iiint_{r<R}\varepsilon_0E^2dV+\frac{1}{2}\iiint_{r>R+t}\varepsilon_0E^2dV.$$

$$\Delta W=W_1-W_0=-\frac{1}{2}\int_R^{R+t}\varepsilon_0E^2\cdot4\pi r^2dr$$

$$=-\frac{q^2}{8\pi\varepsilon_0}\int_R^{R+t}\frac{1}{r^2}dr=\frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{R+t}-\frac{1}{R}\right)<0$$

例. 平行板电容器极板间充满 ε 的介质, 面积 S 间距 d. 则在极板上带 $+Q_0$ 和 $-Q_0$ 的电荷, 外界需要抵抗静电力做多少功?



$$DS=\sigma S.\quad D=\sigma=\frac{Q_0}{S}.$$

$$E=\frac{D}{\varepsilon}=\frac{Q_0}{\varepsilon S}$$

$$W=\frac{1}{2}\varepsilon E^2Sd=\frac{1}{2}\varepsilon\frac{Q_0^2}{\varepsilon^2S^2}Sd=\frac{Q_0^2d}{2\varepsilon S}$$

$$V=Ed=\frac{Q_0}{C},\quad C=\frac{\varepsilon S}{d}.$$

极板带电 kQ_0 时 $(0\leq k\leq1)$.

之间为电势 $k\varphi$, 电势差 $kV=k(\varphi_1-\varphi_2)$

将电荷 Q_0dk 移动,

外界做功 $\delta A=Q_0dk\times kV=kdkQ_0V$

∴将电容器电量从 $0\to Q_0$,

$$A=Q_0V\int_0^1kdk=\frac{1}{2}Q_0V=\frac{1}{2}Q_0\varphi_1+\frac{1}{2}(-Q_0)\varphi_2.$$

$$=\frac{1}{2}(DS)(Ed)=\left(\frac{1}{2}DE\right)Sd=\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right)V.$$