例题

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一、叠加原理

例. 在原点O及$(\sqrt{3}m, 0)$处分别放置$Q_1 = -2\mu C$和$Q_2 = +1\mu C$的点电荷，求$P(\sqrt{3}m, -1m)$处的电场强度。

例. 求均匀带电圆盘轴线上的电场强度。圆盘半径为R，电荷面密度为σ。

① 电偶极子。

② 均匀带电细棒

③ 均匀带电圆环轴线

④ 均匀带电圆盘

二、高斯定理

例. 电荷均匀分布于一个无限大平面上，面密度为σ，求其所激发的静电场的电场强度。

推论：电场强度。

例. 电荷q均匀分布于半径为R的球面上，求球内外的电场强度。

例. 电荷q均匀分布在半径为R的球体上，求球内外的电场强度。

例. 求线电荷密度为ρ_l的均匀带电无限长细棒产生的电场。

例. 求体电荷密度为ρ_v，半径为R的均匀带电球体的电场分布。

例. 如图为带空腔的均匀带电球，其电荷密度ρ，球心到空腔中心的距离为a，求空腔中的场强。

三、计算电势

例. 求均匀带电圆盘轴线上的电势

例. 求均匀带电q球面内外的电势。

③ 典型举例

> 带电圆环轴线

> 带电圆盘

④ 电场 => 电势

例. 均匀电场E_0中放入q，求空间U。

> 无限长均匀带电棒

> 均匀带电球面

> 均匀带电球体

3. 等势面

① 电场与等势面

电势的等值面

电场线与等势面相交一次

$d\varphi = \varphi(\vec r + d\vec l) - \varphi(\vec r) = \nabla\varphi \cdot d\vec l$

$\Rightarrow d\varphi = \int_{\vec r}^{\vec r + d\vec l} \vec E \cdot d\vec l = -\vec E \cdot d\vec l,\ \vec E = -\nabla\varphi$

$\vec E = -\frac{\partial\varphi}{\partial n}\hat n$

② 电偶极子的电势

* 电偶极子

基本：电场 {中垂线上，延长线上，任意位置}　电势

$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r^2+\frac{l^2}{4})} \times 2 \frac{\frac{l}{2}}{\sqrt{r^2+\frac{l^2}{4}}} = \frac{ql}{4\pi\varepsilon_0}\frac{1}{(r^2+\frac{l^2}{4})^{\frac{3}{2}}}$

$x = \frac{\frac{l}{2}}{r} \gg 1$，$E = \frac{ql}{4\pi\varepsilon_0}\frac{1}{(r^2(1+(\frac{l}{2r})^2))^{\frac{3}{2}}} = \frac{ql}{4\pi\varepsilon_0 r^3}(1+(\frac{l}{2r})^2)^{-\frac{3}{2}} = \frac{p}{4\pi\varepsilon_0 r^3} \times \frac{1}{1+\frac{3}{2}\times\frac{l^2}{4r^2}+\cdots} \approx \frac{p}{4\pi\varepsilon_0 r(r^2+\frac{3}{8}l^2)}$

$E(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(r-\frac{l}{2})^2} - \frac{q}{(r+\frac{l}{2})^2}\right] = \frac{q}{4\pi\varepsilon_0}\frac{2rl}{(r-\frac{l}{2})^2(r+\frac{l}{2})^2} \approx \frac{ql}{2\pi\varepsilon_0 r^3}\frac{1}{(1-\frac{l}{2r})^2} \approx \frac{p}{2\pi\varepsilon_0 r^3}$

$\Rightarrow E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$，$E_r = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}$

1. $+q$，$-q$，$-q$，$+q$，$A(x, 0)$

$E_x = \frac{q}{4\pi\varepsilon_0}\left(2\times\frac{1}{(x-\frac{l}{2})^2+(\frac{l}{2})^2}\times\frac{\frac{l}{2}}{\sqrt{(x-\frac{l}{2})^2+(\frac{l}{2})^2}} + 2\times\frac{\frac{l}{2}}{((x+\frac{l}{2})^2+(\frac{l}{2})^2)^{\frac{3}{2}}}\right)$

$= \frac{q}{4\pi\varepsilon_0}\left(\frac{l}{(x^2+\frac{l^2}{2}-xl)^{\frac{3}{2}}} - \frac{l}{(x^2+\frac{l^2}{2}+xl)^{\frac{3}{2}}}\right)$

$x \gg l$时，$\approx \frac{1}{(x^2-xl)^{\frac{3}{2}}} - \frac{1}{(x^2+xl)^{\frac{3}{2}}} = \frac{1}{x^3}\left[\frac{1}{(1-\frac{l}{x})^{\frac{3}{2}}} - \frac{1}{(1+\frac{l}{x})^{\frac{3}{2}}}\right] \approx \frac{1}{x^3}(1+\frac{3l}{2x}-1+\frac{3l}{2x}) = \frac{3l}{x^3}$

$(1+x)^n \approx 1+nx$　$\frac{1}{\sqrt{1+x}} = 1-\frac{1}{2}x+\frac{3}{8}x^2$

$(t+x)^n = t^n(1+\frac{x}{t})^n \approx t^n(1+\frac{nx}{t})$. 保留最低阶那several项.

2. 受力&力矩.

3. $+q$，$-2q$，$+q$

$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1}+\frac{q}{r_2}-\frac{2q}{r}\right)$　$r^2+l^2-2lr\cos\theta = r_1^2$

$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+l^2-2lr\cos\theta}} + \frac{1}{\sqrt{r^2+l^2+2lr\cos\theta}} - \frac{2}{r}\right)$

$= \frac{q}{4\pi\varepsilon_0 r}\left(\frac{1}{\sqrt{1+(\frac{l}{r})^2-2(\frac{l}{r})\cos\theta}} + \frac{1}{\sqrt{1+(\frac{l}{r})^2+2(\frac{l}{r})\cos\theta}} - 2\right)$

$= \frac{q}{4\pi\varepsilon_0 r}\left[1-\frac{1}{2}(\frac{l^2}{r^2}-\frac{2l}{r}\cos\theta)+\frac{3}{8}(\frac{l^2}{r^2}-\frac{2l}{r}\cos\theta)^2 + 1-\frac{1}{2}(\frac{l^2}{r^2}+\frac{2l}{r}\cos\theta)+\frac{3}{8}(\frac{l^2}{r^2}+\frac{2l}{r}\cos\theta)^2\right]$

$= \frac{q}{4\pi\varepsilon_0 r}\left(-\frac{l^2}{r^2}+\frac{3l^2}{r^2}\cos^2\theta\right) \approx \frac{ql^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta-1)$

$\vec E = -\nabla U = -\frac{\partial U}{\partial r}\vec e_r - \frac{1}{r}\frac{\partial U}{\partial\theta}\vec e_\theta = \frac{ql^2}{4\pi\varepsilon_0}\left(\frac{3}{r^4}(3\cos^2\theta-1)+\frac{6\cos\theta\sin\theta}{r^4}\right)$

$= \frac{3ql^2}{4\pi\varepsilon_0 r^4}((3\cos^2\theta-1)\vec e_r + 2\sin\theta\cos\theta\,\vec e_\theta)$

例. 在原点O及$(\sqrt{3}m, 0)$处分别放置$Q_1 = -2\mu C$和$Q_2 = +1\mu C$的点电荷，求$P(\sqrt{3}m, -1m)$处的电场强度。

$\vec E_1 = k\frac{Q_1}{|PO|^2}\hat{PO}$

$\vec E = \vec E_1 + \vec E_2$

例. 求均匀带电圆盘轴线上的电场强度。圆盘半径为R，电荷面密度为σ。

半径为r，r+dr的圆，表面dq的半径.

$dE_z = k\frac{dq}{z^2+r^2}\cos\alpha$

$= \frac{\sigma}{2\varepsilon_0}\left[1-\frac{1}{\sqrt{1+(\frac{R}{z})^2}}\right]$

① 电偶极子。

相隔一定距离的等量异号电荷。

电偶极矩$p = ql$.　$+q \quad l \quad -q$

> 中垂线上。

$E = 2\times\frac{q}{4\pi\varepsilon_0}\times\frac{1}{(r^2+\frac{l^2}{4})}\times\frac{\frac{l}{2}}{\sqrt{r^2+\frac{l^2}{4}}}$

$= \frac{ql}{4\pi\varepsilon_0}\frac{1}{(r^2+\frac{l^2}{4})^{\frac{3}{2}}} = \frac{ql}{4\pi\varepsilon_0}\frac{1}{(r^2(1+\frac{l^2}{4r^2}))^{\frac{3}{2}}}$

令$x = \frac{\frac{l}{2}}{r} \ll 1$，$(1+x)^{\frac{3}{2}} \approx 1+\frac{3}{2}x^2+\cdots \approx 1$.

$\approx \frac{ql}{4\pi\varepsilon_0}\times\frac{1}{r^3} = \frac{p}{4\pi\varepsilon_0 r^3}$.

> 延长线上。

$E = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(r-\frac{l}{2})^2} - \frac{1}{(r+\frac{l}{2})^2}\right)$

$= \frac{q}{4\pi\varepsilon_0}\frac{(r+\frac{l}{2})^2-(r-\frac{l}{2})^2}{(r-\frac{l}{2})^2(r+\frac{l}{2})^2}$

$= \frac{q}{4\pi\varepsilon_0}\frac{2rl}{r^4(1-\frac{l}{2r})^2(1+\frac{l}{2r})^2}$

$\frac{l}{2r} \ll 1$，$= \frac{ql}{2\pi\varepsilon_0 r^3}\frac{1}{(1-\frac{l^2}{4r^2})} \approx \frac{p}{2\pi\varepsilon_0 r^3}$.

> 任意位置，矢量分解。

$E_{//} = \frac{p}{2\pi\varepsilon_0(r\cos\theta)^3}$

$E_\perp = \frac{p}{4\pi\varepsilon_0(r\sin\theta)^3}$.

看作两个电偶极子

$p_{//} = p\cos\theta$，$p_\perp = p\sin\theta$.

$E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$，$E_r = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}$

$\vec E = \frac{3\hat n(\vec p\cdot\hat n)-\vec p}{4\pi\varepsilon_0 r^3}$.

② 均匀带电细棒

$\lambda = \frac{Q}{L}$，$dQ = \lambda dy = \frac{Q}{L}dy$.

$d\vec E = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\hat R = \frac{\lambda}{4\pi\varepsilon_0}\frac{dy}{x^2+(y-y')^2}\hat R$

$\begin{cases} y-y' = x\cot\theta \\ x = R\sin\theta \end{cases}$　$dy = \frac{s}{\sin^2\theta}d\theta = \frac{R^2}{s}d\theta$.

$\vec E = \frac{\lambda}{4\pi\varepsilon_0 s}\int_{\theta_1}^{\theta_2}\frac{\frac{R^2}{s}d\theta}{R^2\sin^2\theta+R^2\sin^2\theta\cot^2\theta}\hat R$

$= \frac{\lambda}{4\pi\varepsilon_0 s}\int_{\theta_1}^{\theta_2}\hat R\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 s}\int_{\theta_1}^{\theta_2}(\hat x\sin\theta+\hat y\cos\theta)\,d\theta$

$= \frac{\lambda}{4\pi\varepsilon_0 s}(\hat y\sin\theta - \hat x\cos\theta)\Big|_{\theta_1}^{\theta_2}$

$= \frac{\lambda}{4\pi\varepsilon_0 s}[\hat y(\sin\theta_2-\sin\theta_1) - \hat x(\cos\theta_2-\cos\theta_1)]$

$|\vec E| = \frac{\lambda}{2\pi\varepsilon_0 s}\sin\frac{\theta_2-\theta_1}{2}$.

方向：角平分线。

(等效为圆弧)

无限长 $\Rightarrow \theta_2\to\pi,\ \theta_1\to 0$.

$|\vec E| = \frac{\lambda}{2\pi\varepsilon_0 s}$，方向垂直于棒。

③ 均匀带电圆环轴线。

$d\vec E = \frac{\lambda dl}{4\pi\varepsilon_0(s^2+R^2)}\frac{s}{\sqrt{s^2+R^2}}$，$dl = Rd\theta$

$\vec E = \int_0^{2\pi}\frac{\lambda R}{4\pi\varepsilon_0}\frac{s\,d\theta}{(s^2+R^2)^{\frac{3}{2}}} = \frac{\lambda Rs}{2\varepsilon_0(s^2+R^2)^{\frac{3}{2}}}\hat z$.

$\frac{\lambda R}{2\varepsilon_0}\frac{(s^2+R^2)^{\frac{3}{2}} - s\cdot 2s^2(s^2+R^2)^{\frac{1}{2}}}{(s^2+R^2)^3}$

$\frac{R}{\sqrt{2}}$，$\frac{\lambda}{3\sqrt{3}\varepsilon_0 a}$

④ 均匀带电圆盘。

$\lambda = \frac{Q}{2\pi R}$，$\vec E_{环} = \frac{\lambda Rs}{2\varepsilon_0(s^2+R^2)^{\frac{3}{2}}} = \frac{Qs}{4\pi\varepsilon_0(s^2+R^2)^{\frac{3}{2}}}$　环电量

$d\vec E = dQ\frac{s}{4\pi\varepsilon_0(s^2+r^2)^{\frac{3}{2}}} = \sigma dS\frac{s}{4\pi\varepsilon_0(r^2+s^2)^{\frac{3}{2}}}$

$= \frac{\sigma 2\pi r\,dr\,s}{4\pi\varepsilon_0(r^2+s^2)^{\frac{3}{2}}} = \frac{\sigma sr}{2\varepsilon_0(r^2+s^2)^{\frac{3}{2}}}dr$

$\vec E = \int_0^R\frac{\sigma sr}{2\varepsilon_0(r^2+s^2)^{\frac{3}{2}}}dr = \frac{\sigma s}{\varepsilon_0}\int\frac{dr^2}{(r^2+s^2)^{\frac{3}{2}}} = \frac{\sigma s}{\varepsilon_0}\frac{1}{(r^2+s^2)^{\frac{1}{2}}}\Big|_0^R$

$= \frac{\sigma s}{2\varepsilon_0}\left(\frac{1}{s} - \frac{1}{\sqrt{s^2+R^2}}\right)$

若$s \gg R$，$\vec E = \hat z\frac{\sigma}{2\varepsilon_0}\left[\frac{s}{|s|} - (1+\frac{R^2}{s^2})^{-\frac{1}{2}}\right]$

$= \hat z\frac{\sigma}{2\varepsilon_0}\left[1-(1-\frac{R^2}{2s^2})\right] = \hat z\frac{\sigma R^2}{4\varepsilon_0 s^2}$

即$\vec E = \frac{Q}{4\pi\varepsilon_0 s^2}\hat z$.　点电荷。

若$R \gg s$，$\lim_{R\to+\infty}\frac{1}{\sqrt{R^2+s^2}} = 0$，

$\vec E = \hat z\frac{\sigma}{2\varepsilon_0}\frac{s}{|s|} = \begin{cases} +\frac{\sigma}{2\varepsilon_0}\hat z, & z>0 \\ -\frac{\sigma}{2\varepsilon_0}\hat z, & z<0 \end{cases}$　平面。

例. 电荷均匀分布于一个无限大平面上，面密度为σ，求其所激发的静电场的电场强度。

① 假设$\vec E$不与面垂直

$\vec E$沿该面的法向量$\vec n$，则$\vec E$有切向分量$\vec E_{//}$.

将$\vec E$绕轴线旋转180°，会使场强改变但面无限大，面的旋转不改变$\vec E$分布.

∴ $\vec E$与面垂直.

② $\oint\vec E\cdot d\vec S = 2ES = \frac{\sigma S}{\varepsilon_0}$，$dq = \sigma dS$.

∴ $E = \frac{\sigma}{2\varepsilon_0}$

加上方向. $\vec E = \frac{\sigma}{2\varepsilon_0}\hat e_n$

推论：电场强度。

$E = \frac{\sigma}{\varepsilon_0}$

例. 电荷q均匀分布于半径为R的球面上，求球内外的电场强度。

外：作半径为r的球面.

$\oint\vec E\cdot d\vec S = \oint E\,dS = E\cdot 4\pi r^2$

又$\oint = \frac{q}{\varepsilon_0}$. ∴ $\vec E = \frac{q}{4\pi\varepsilon_0 r^2}\hat e_r$

内：作半径为r'的球面.

例. 电荷q均匀分布在半径为R的球体上，求球内外的电场强度。

外：$\oint\vec E\cdot d\vec S = \oint E\,dS = E\cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$.

内：$\oint\vec E'\cdot d\vec S = E\cdot 4\pi r^2 = (\frac{r}{R})^3\frac{q}{\varepsilon_0}$

$E = \frac{qr}{4\pi R^3\varepsilon_0}$

例. 求线电荷密度为ρ_l的均匀带电无限长细棒产生的电场。

$\Phi = \oint\vec E\cdot d\vec S = \oint E\,dS = E\cdot 2\pi rL = \frac{\rho_l L}{\varepsilon_0}$

$\therefore E = \frac{\rho_l}{2\pi r\varepsilon_0}\hat e_r$

例. 求体电荷密度为ρ_v，半径为R的均匀带电球体的电场分布。

$r>R$，$\Phi = \oint\vec E\cdot d\vec S = E\cdot 4\pi r^2 = \frac{\frac{4}{3}\pi R^3\rho}{\varepsilon_0}$

$\therefore E = \frac{\rho R^3}{3\varepsilon_0 r^2}$

$r<R$，$\Phi = \oint\vec E\cdot d\vec S = E\cdot 4\pi r^2 = \frac{\frac{4}{3}\pi\rho r^3}{\varepsilon_0}$

$\therefore E = \frac{\rho r}{3\varepsilon_0}$

例. 如图为带空腔的均匀带电球，其电荷密度ρ，球心到空腔中心的距离为a，求空腔中的场强。

看作$+\rho$大球与$-\rho$小球的叠加.

$\vec E_1 = \frac{\rho\vec r}{3\varepsilon_0}$，$\vec E_2 = -\frac{\rho\vec r'}{3\varepsilon_0}$

$\vec E = \vec E_1 + \vec E_2 = \frac{\rho}{3\varepsilon_0}(\vec r - \vec r') = \frac{\rho}{3\varepsilon_0}\vec a$

例. 求均匀带电圆盘轴线上的电势

$r\to dr$　$dS = r\,dr\,d\varphi$.　$dq = \sigma r\,dr\,d\varphi$.

$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2+z^2}$，$dU = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{r^2+z^2}} = \frac{\sigma\,r\,dr\,d\varphi}{4\pi\varepsilon_0\sqrt{r^2+z^2}}$

$V = \int_0^{2\pi}d\varphi\int_0^R\frac{\sigma}{4\pi\varepsilon_0}\frac{r\,dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\int_0^R\frac{r\,dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2}-z)$

例. 求均匀带电q球面内外的电势。

$\rho = \frac{q}{\frac{4}{3}\pi R^3}$.　$d\tau = r\,d\varphi\,r\cos\varphi\,d\theta\,dr$

$V = \frac{1}{4\pi\varepsilon_0}\int_0^R\int_0^\pi\int_0^{2\pi}\rho r^2 d\varphi\cos\varphi\,d\theta\,dr$

$= \frac{1}{4\pi\varepsilon_0}\times 2\pi\times\frac{q}{\frac{4}{3}\pi R^3}\int_0^R\int_0^\pi r^2\cos\varphi\,d\varphi\,dr$

$= \frac{3q}{4\pi R^3\varepsilon_0}\frac{1}{3}R^3 = \frac{q}{4\pi\varepsilon_0 r}$.

电场强度 $\vec E = \begin{cases}\frac{q}{4\pi\varepsilon_0 r^2}\vec e & r>R \\ 0 & r<R\end{cases}$

球外任意取A，距O为r.

$V = \int_A^{P_0}\vec E\cdot d\vec l$，$P_0$为无限远，则可看作点电荷

$V = \frac{q}{4\pi\varepsilon_0 r}$.

球内取B，$V = \int_B^C\vec E\cdot d\vec l + \int_C^{P_0}\vec E\cdot d\vec l = 0 + \int_R^\infty\frac{q}{4\pi\varepsilon_0}\frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0 R}$.

③ 典型举例

> 带电圆环轴线

$\varphi(z) = \int_0^{2\pi}\frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,d\theta}{\sqrt{R^2+z^2}} = \frac{\lambda R}{2\varepsilon_0\sqrt{R^2+z^2}} = \frac{q}{4\pi\varepsilon_0\sqrt{R^2+z^2}}$

> 带电圆盘

$\varphi = \frac{q}{4\pi\varepsilon_0\sqrt{R^2+z^2}} \xrightarrow[=\sigma 2\pi r dr]{dq=\sigma dS} d\varphi = \frac{\sigma}{2\varepsilon_0}\frac{r\,dr}{\sqrt{r^2+z^2}}$

$\varphi = \int_0^R\frac{\sigma}{2\varepsilon_0}\frac{r\,dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2}-|z|)$

④ 电场 => 电势

例. 均匀电场E_0中放入q，求空间U.

P点电势为0.

$\varphi_1(\vec r) = -\int_P^{\vec r}\vec E\cdot d\vec l = -\vec E_0\cdot\vec r + C_1$

$\varphi_2(\vec r) = \frac{q}{4\pi\varepsilon_0 r} + C_2$.

$\varphi = \varphi_1 + \varphi_2 = -\vec E_0\cdot\vec r + \frac{q}{4\pi\varepsilon_0 r} + C$.

$C = C_1 + C_2$，待定.

> 无限长均匀带电棒

$E\cdot 2\pi rl = \frac{\lambda l}{\varepsilon_0}$，$\vec E = \frac{\lambda}{2\pi\varepsilon_0 r}\hat r$

$\varphi = -\int_P^{\vec r}\vec E\cdot d\vec l = -\int_P^a\vec E\cdot d\vec l - \int_a^{\vec r}\vec E\cdot d\vec l$

$= -\int_a^r\frac{\lambda}{2\pi\varepsilon_0 r}dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C$.

> 均匀带电球面

$\vec E = \begin{cases}\frac{q}{4\pi\varepsilon_0 r^2}, & r>R \\ 0, & r<R\end{cases}$

$\varphi = \begin{cases}\frac{q}{4\pi\varepsilon_0 r}, & r>R \\ \frac{q}{4\pi\varepsilon_0 R}, & r<R\end{cases}$

> 均匀带电球体

$\vec E = \begin{cases}\frac{q\hat r}{4\pi\varepsilon_0 r^2}, & r>R \\ \frac{q\vec r}{4\pi\varepsilon_0 R^3}, & r<R\end{cases}$　$\varphi = \frac{q}{4\pi\varepsilon_0 r}$

$\varphi = -\int_\infty^r\vec E\cdot d\vec l = -\int_\infty^R\vec E\cdot d\vec l - \int_R^r\vec E\cdot d\vec l$

$= \varphi(R) - \frac{q}{4\pi\varepsilon_0 R^3}\int_R^r\vec r\cdot d\vec r$

$= \frac{q}{4\pi\varepsilon_0 R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$.

3. 等势面.

① 电场与等势面.

电势的等值面.

电场线与等势面相交一次.

$d\varphi = \varphi(\vec r + d\vec l) - \varphi(\vec r) = \nabla\varphi\cdot d\vec l$.

$\Rightarrow d\varphi = \int_{\vec r}^{\vec r+d\vec l}\vec E\cdot d\vec l = -\vec E\cdot d\vec l$，$\vec E = -\nabla\varphi$.

$\vec E = -\frac{\partial\varphi}{\partial n}\hat n$.

② 电偶极子的电势

$\varphi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$

$= \frac{q}{4\pi\varepsilon_0}\frac{(r_- - r_+)(r_- + r_+)}{r_+ r_-(r_- + r_+)} \approx \frac{q}{4\pi\varepsilon_0}\frac{2\vec l\cdot\vec r}{2r^3}$.

$\vec r_+ = \vec r - \frac{\vec l}{2}$，$\vec r_- = \vec r + \frac{\vec l}{2}$，$r_-^2 - r_+^2 = 2\vec l\cdot\vec r$.

电场 $\vec E = -\nabla\varphi = \hat r\frac{\partial\varphi}{\partial r} + \frac{\hat\theta}{r}\frac{\partial\varphi}{\partial\theta} + \frac{\hat\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}$

$= -\left(-\frac{2\cos\theta}{r^3}\hat r - \frac{\sin\theta}{r^3}\hat\theta\right) = \frac{2p\cos\theta\,\hat r + p\sin\theta\,\hat\theta}{4\pi\varepsilon_0 r^3}$

$= \frac{2(\vec p\cdot\hat r)\hat r - (\vec p\cdot\hat\theta)\hat\theta}{4\pi\varepsilon_0 r^3} = \frac{3(\vec p\cdot\hat r)\hat r - \vec p}{4\pi\varepsilon_0 r^3}$

核外电荷：$\rho(r) = \frac{-qe^{-\frac{2r}{a}}}{\pi a^3}$

S_1 r处. $E\times 4\pi r^2 = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_0^r\frac{-q}{\pi a^3}e^{-\frac{2r}{a}}4\pi r^2 dr$

$= \frac{-4q}{a^3\varepsilon_0}\int_0^r e^{-\frac{2r}{a}}r^2 dr$

$= \frac{-4q}{a^3\varepsilon_0}\left(\frac{r^2}{-\frac{2}{a}} - \frac{r}{2(\frac{2}{a})^2} + \frac{1}{2(-\frac{2}{a})^3}\right)e^{-\frac{2r}{a}}$

$= \frac{q}{\varepsilon_0}\left(\frac{2r^2}{a^2} + \frac{r}{2a} + \frac{1}{4}\right)e^{-\frac{2}{a}r}$

$E = \frac{4q}{\varepsilon_0}($

$\int e^{kr}r^2 dr$

$= \frac{1}{k}\int r^2 de^{kr} = \frac{1}{k}(r^2e^{kr} - \int e^{kr}dr^2) = \frac{1}{k}(r^2e^{kr} - \frac{1}{2}\int e^{kr}r\,dr)$

$= \frac{1}{k}r^2e^{kr} - \frac{1}{2k}\times\frac{1}{k}\int r\,de^{kr} = \frac{r^2}{k}e^{kr} - \frac{1}{2k^2}(re^{kr} - \int e^{kr}dr)$

$= \frac{r^2}{k}e^{kr} - \frac{r}{2k^2}e^{kr} + \frac{1}{2k^3}e^{kr}$.

S_2 $dq = \rho(R)dV = 2\pi\rho(R)R^2\sin\theta\,dR\,d\theta$

$dU = \frac{dq}{4\pi\varepsilon_0\sqrt{R^2+r^2-2Rr\cos\theta}} = \frac{1}{2\varepsilon_0}\frac{\rho(R)R^2\sin\theta\,dR\,d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}}$

$U = \frac{1}{2\varepsilon_0}\int_0^\infty\int_0^\pi\rho(R)R^2\frac{\sin\theta\,d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}}dR$　分段

$= \frac{1}{\varepsilon_0 r}\int_0^r\rho(R)R^2dR + \frac{1}{\varepsilon_0}\int_r^\infty\rho(R)R\,dR$.

全空间所有电荷. 对r处.

$E_L S + E_r S = \int_0^b kx S\,dx$

$E_L + E_r = \frac{1}{\varepsilon_0}\frac{1}{2}kb^2 \Rightarrow E = \frac{kb^2}{4\varepsilon_0}\vec e_x$

$E S + E_L S = \frac{1}{\varepsilon_0}\int_x^b kxS\,dx$

$E = \frac{1}{\varepsilon_0}\frac{k}{2}(b^2-x^2) - \frac{kb^2}{4\varepsilon_0} = \frac{kb^2}{4\varepsilon_0} - \frac{kx^2}{2\varepsilon_0}$

$\vec E = \int_0^x\frac{kx\,dx}{2\varepsilon_0}\vec e_x - \int_x^b\frac{kx\,dx}{2\varepsilon_0}\vec e_x$